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A CONTINUOUS DEPENDENCE OF FIXED POINTS OF
\(\phi\)-CONTRACTIVE MAPPINGS IN UNIFORM SPACES

Vasil G. Angelov

Abstract. The main purpose of the present paper is to establish conditions for a continuous dependence of fixed points of \(\phi\)-contractive mappings in uniform spaces. An application to nonlinear functional differential equations of neutral type have been made.

The main purpose of the present paper is to establish when the convergence of a sequence of \(\phi\)-contractive mappings in a uniform space implies a convergence of the sequence of their fixed points. The notion a \(\phi\)-contractive mapping in a uniform space has been introduced in [1]. In view of the applications given in [1] the problem of a continuous dependence of fixed points can be formulated as a continuous dependence of the solutions of a nonlinear functional differential equation on its right-hand side. As a particular case we obtain an extension of the results from [2] and [3] in metric spaces.

Since [4] contains the most general version of fixed point for \(\phi\)-contractive mappings in uniform spaces we shall recall some basic definitions and results from [4].

Let \((X, \mathcal{A})\) be a complete Hausdorff uniform space with a uniformity generated by saturated family of pseudometrics \(\mathcal{A} = \{d_{\alpha}(x, y) : \alpha \in \mathcal{A}\}\), \(\mathcal{A}\) being an index set (cf [5]) Let \(j : A \to A\) be a mapping and let \(j^k(\alpha) = j(j^{k-1}(\alpha)), j^0(\alpha) = = \alpha, (k = 1, 2, 3, \ldots)\). Since \(j\) is not image of the element \(\alpha_0 \in \mathcal{A}\), that is \(j^{-1}(\alpha_0) = = \{\alpha \in A : j(\alpha) = \alpha_0\}\). In general we define

\[
j^{-n}(\alpha_0) = \{\alpha \in A : j^n(\alpha) = \alpha_0\} \quad (n = 2, 3, \ldots) .
\]

The space \(X\) is called \(j\)-bounded if for every \(x, j \in X\) and \(\alpha \in \mathcal{A}\) there exists a constant \(Q = Q(\alpha, x, y) > 0\) such that

\[
d_{j^{-n}(\alpha)}(x, y) \leq Q \quad (\alpha, x, y) < \infty \quad (n = 0, 1, 2, \ldots) .
\]

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The last inequality is in the sense defined in [4]. Further on we shall assume that 
$X$ is a $j$-bounded space.

Let $(\phi)$ be a family of contractive functions $\Phi_\alpha(t)R^1_+ \to R^1_+$, $R^1_+ = [0, \infty)$, $\alpha \in A$ with the properties:

\[
\Phi_\alpha(t) \quad \text{is strictly increasing, continuous from the right,}
0 < \Phi_\alpha(t) < t \quad \text{and } \Phi_{j(\alpha)}(t) \leq \Phi_\alpha(t) \quad \text{for } t \geq 0 \quad \text{and}
\]

(\Phi 1) \quad \Phi_\alpha(t_1 + t_2) \leq \Phi_\alpha(t_1) + \Phi_\alpha(t_2) \quad \text{for every } t_1, t_2 > 0.

\[\Phi_\alpha(t) \quad \text{is strictly increasing, continuous from the right,}
0 < \Phi_\alpha(t) < t \quad \text{and } \Phi_{j(\alpha)}(t) \leq \Phi_\alpha(t) \quad \text{for } t \geq 0 \quad \text{and}
\]

(\Phi 2) \quad \lim_{n \to \infty} \Phi_\alpha(\Phi_{j-1(\alpha)}(\ldots \Phi_{j-n(\alpha)}(t) \ldots)) = 0

which ought to be understand in the following sense: for every sequence $\alpha, \alpha_1, \ldots, \alpha_n, \ldots (\alpha_n \in j^{-n}(\alpha)) \lim_{n \to \infty} \Phi_\alpha(\Phi_{\alpha_1}(\ldots \Phi_{\alpha_n}(t) \ldots)) = 0$.

The mapping $T$ is called: 1) $\phi$-contractive if $d_{j(\alpha)}(Tx, Ty) \leq \Phi_\alpha(d_\alpha(x, y))$ for every $x, y \in X$ and $\alpha \in A$; 2) contractive if $d_{j(\alpha)}(Tx, Ty) < d_\alpha(x, y)$ for every $x, y \in X$ and $\alpha \in A$; 3) $j$-regular when if $\{T^n x\}_{n=0}^\infty$ is not $d_\alpha$-Cauchy sequence, then it is not $d_{j(\alpha)}$-Cauchy sequence for every $x \in (\text{or equivalently, if} \{T^n x\}_{n=0}^{\infty})$ is $d_{j(\alpha)}$-Cauchy sequence, then it is $d_\alpha$-Cauchy sequence).

Theorem A. [4] Every $\phi$-contractive $j$-regular mapping $T : X \to X$ has a unique fixed point $\bar{x} \in X$ and $\bar{x} = \lim_{n \to \infty} T^n x$ for arbitrary $x \in X$.

Main results

Let $\{T_k\}_{k=1}^\infty$ be a sequence of operators $T_k : X \to X$ . Every $T_k$ has at least one fixed point $y_k (k = 1, 2, 3, \ldots)$. Let $T_0 : X \to X$ be a $\phi$-contractive $j$-regular mapping with fixed point $y_0$. We say that the sequence $\{T_k\}_{k=1}^\infty$ tends uniformly to $T_0$ if for every $\varepsilon > 0$ there exists $\nu = \nu(\varepsilon)$ such that $d_\alpha(T_k x, T_0 x) < \varepsilon$ for every $k > \nu, x \in X, \alpha \in A$.

Theorem 1. If the sequence $\{T_k\}_{k=1}^\infty$ converges uniformly to $T_0$, then the sequence $\{y_k\}_{k=1}^\infty$ converges to $y_0$.

Proof. In view of the uniform convergence of the sequence $\{T_k\}_{k=1}^\infty$ to $T_0$ for every $\varepsilon > 0$, $\alpha \in A$ there is $\nu_1$ such that $d_\alpha(T_k y, T_0 y) < \varepsilon/2$ for every $y \in X$.

So that we have for $k > \nu_1$

\[
d_\alpha(y_k, y_0) = d_\alpha(T_k y_k, T_0 y_0) \leq \frac{\varepsilon}{2} + \Phi_{j-1(\alpha)}(d_{j-1(\alpha)}(y_k, y_0))
\]

For $\varepsilon/2^2$ we find $\nu_2$ such that when $k > \nu_2$ we have

\[
d_{j-1(\alpha)}(y_k, y_0) \leq \frac{\varepsilon}{2^2} + \Phi_{j-2(\alpha)}(d_{j-2(\alpha)}(y_k, y_0))
\]
Therefore for $k > \max \{ \nu_1, \nu_2 \}$ the following inequalities are fulfilled:

$$
d_{\alpha}(y_k, y_0) \leq \frac{\varepsilon}{2} + \Phi_{j^{-1}(\alpha)}(\frac{\varepsilon}{2^n} + \Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0))) \leq
\leq \frac{\varepsilon}{2} + \Phi_{j^{-1}(\alpha)}(\frac{\varepsilon}{2^n}) + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0))) \leq
\leq \frac{\varepsilon}{2} + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(d_{j^{-2}(\alpha)}(y_k, y_0)))
$$

We can proceed in an analogous way and then obtain for $k > N_n = \max \{ \nu_1, \nu_2, \ldots, \nu_n \}$

$$
d_{\alpha}(y_k, y_0) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^n} + \cdots + \frac{\varepsilon}{2^n} + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\cdots \Phi_{j^{-n+1}(\alpha)}(d_{j^{-n+1}(\alpha)}(y_k, y_0))) \cdots) \leq
\leq \varepsilon + \Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\cdots \Phi_{j^{-n+1}(\alpha)}(d_{j^{-n+1}(\alpha)}(y_k, y_0))) \cdots))
$$

Let us fix $n$ sufficiently large such that

$$
\Phi_{j^{-1}(\alpha)}(\Phi_{j^{-2}(\alpha)}(\cdots \Phi_{j^{-n+1}(\alpha)}(d_{j^{-n+1}(\alpha)}(y_k, y_0))) \cdots) < \varepsilon
$$

Then for $k > N_n$ we have $d_{\alpha}(y_k, y_0) < 2\varepsilon$. Theorem 1 is thus proved. \hfill \Box

**Remark 1.** If we replace the definition of $\phi$-contractive mapping by the following one: $d_{\alpha}(T_x, T_y) \leq \Phi_{\alpha}(d_{j(\alpha)}(x, y))$ then the assertion of Theorem 1 is also valid. We must only modify conditions (\Phi 1) and (\Phi 2), namely, $\Phi_{\alpha}(t) \leq \Phi_{j(\alpha)}(t)$ and

$$
\lim_{n \to \infty} \Phi_{\alpha}(\Phi_{j(\alpha)}(\cdots \Phi_{j^{n+1}(\alpha)}(t) \cdots)) = 0
$$

The definition of a $j$-bounded and $j$-regular mapping can be modified in an obvious way.

**Remark 2.** Let $X$ be a quasicomplete uniform space. This means that every closed bounded subset of $X$ is complete in the induced topology. Consequently if $T : M \to M$ is a $\phi$-contractive mapping of a bounded closed set $M \subset X$ into itself, then $T$ has a unique fixed point in $M$ (cf. [1], Theorem 1).

Further on we shall assume that $X$ is a locally compact space. Let us recall the relations between locally compact spaces and uniformizable spaces (cf. [6]). Every completely regular $T_1$-space is said to be a Tikhonoff’s one. It is known (cf. [6]) that every locally compact Hausdorff space is a Tikhonoff’s space. On the other hand $X$ is uniformizable if and only if $X$ is a completely regular space. So that we shall assume that $X$ is a locally compact quasicomplete Hausdorff space. We shall denote again by $A$ its uniformity, that is, $A = \{ d_{\alpha}(x, y) : \alpha \in A \}$ (cf. [5], [6]).
Theorem 2. Let \((X, \mathcal{A})\) be a locally compact Hausdorff quasicomplete \(j\)-bounded uniform space. Let \(T_k : X \to X\) be a \(\phi\)-contractive mapping with fixed point \(y_k\) for any \(k = 0, 1, 2, \ldots\), i.e. \(d_\phi(T_k x, T_k y) < \Phi_\phi(d_{j_{\phi}}(x, y))\). If \(\{T_k\}_{k=1}^\infty\) converges pointwise to \(y_0\), then the sequence \(\{y_k\}_{k=1}^\infty\) converges to \(y_0\).

Proof. Let us choose \(\varepsilon > 0\) and \(\alpha_1, \ldots, \alpha_p \in A\) such that the neighbourhood \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0) = \{x \in X : d_\phi(x, y_0) < \varepsilon\}\) of \(y_0\) is a compact subset of \(X\). The sequence \(\{T_k\}_{k=1}^\infty\) is equicontinuous and converges pointwise to \(T_0\). But \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\) is compact and in view of the results of Ch. VII [6], \(\{T_k\}_{k=1}^\infty\) converges uniformly on \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\) to \(0\). Then let for \(k > N_n = \max \{\nu_1, \ldots, \nu_n\} (s = 1, \ldots, p)\) we have

\[
\begin{align*}
    d_\phi(T_k y, y_0) &= d_\phi(T_k y, T_0 y_0) \leq d_\phi(T_k y, T_0 y) + d_\phi(T_0 y, T_0 y_0) \\
    &\leq \frac{\varepsilon}{2^2} + \Phi_\phi(d_{j_{\phi}}(y, y_0)) \leq \frac{\varepsilon}{2^2} + \Phi_\phi(d_{j_{\phi}}(y, y_0)) \\
    &\leq \frac{\varepsilon}{2^2} + \Phi_\phi\left(\frac{\varepsilon}{2^3} + \Phi_\phi(d_{j_{\phi}}(y, y_0))\right) \leq \ldots \\
    &\leq \frac{\varepsilon}{2^2} + \cdots + \left(\frac{\varepsilon}{2^m} + \Phi_\phi(d_{j_{\phi}}(y, y_0))\right) \leq \frac{\varepsilon}{2} = \varepsilon
\end{align*}
\]

We obtained that \(T_k\) maps \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\) into itself.

Denote by \(T_k/N_\varepsilon\) the restriction of \(T_k\) to \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\). But \(X\) is quasicomplete and \(j\)-bounded. The same properties has and \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\). Then \(T_k/N_\varepsilon\) possesses a fixed point \(\tilde{y}_k\) for \(k > N_n\) in \(N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\). On the other hand \(T_k\) has only one fixed point \(y_k\). Consequently \(\tilde{y}_k = y_k \in N_\varepsilon(\alpha_1, \ldots, \alpha_p)(y_0)\) for \(k > N_n\). So we have \(\lim_{k \to \infty} y_k = y_0\) which completes proof of Theorem 2.

Let \(A_1 = \{d_{\alpha_1}(x, y) : \alpha_1 \in A\}\) be two families of pseudometrics for the same set \(X\). They will be called equivalent if and only if the identity mapping from \((X, A_1)\) to \((X, A_2)\) is a homomorphism. Further on, we shall assume that card \(A_1 = \text{card} A_2\), so that we shall not differ the index sets of equivalent families of pseudometrics.

A sequence of families of pseudometrics \(\{A_n\}_{n=1}^\infty\) tends uniformly to a family \(A_0\) if for every \(\varepsilon > 0\) there is \(N\) such that for every \(n > N, x, y \in X\) and \(\alpha \in A\)

\[
|d_\alpha^n(x, y) - d_\alpha^0(x, y)| < \varepsilon
\]

Proposition 1. Let \(\{A_n\}_{n=1}^\infty\) be a sequence of families of pseudometrics on \(X\) which tends uniformly to the family \(A_0\) such that each \(A_n\) is equivalent to \(A_0\). Let \(\{T_n\}_{n=1}^\infty\) be a sequence of \(\phi\)-contractive mappings converging pointwise on \(X\) to a mapping \(T_0\). Then \(\{T_n\}_{n=1}^\infty\) converges \(A_0\)-uniformly for every compact set \(K \subset X\) to \(T_0\) (\(T_n\) is \(\phi\)-contractive with respect to the family \(A_n\)).
Proof. For an arbitrary $\varepsilon > 0$ we choose $\eta = \frac{\varepsilon}{3}$. Let for $n > N$, $\alpha \in A$ and $x, y \in X$, $\bar{\alpha} \in j^{-1}(\alpha)$ the inequality holds

$$|d_{\bar{\alpha}}^{(n)}(x, y) - d_{\bar{\alpha}}^{(0)}(x, y)| < \eta.$$ 

If $n > N$ and $x, y \in X$ for which $d_{\bar{\alpha}}^{(0)}(x, y) < \eta$ then

$$d_{\bar{\alpha}}^{(0)}(T_n x, T_n y) < \eta + d_{\bar{\alpha}}^{(n)}(T_n x, T_n y) < \eta + d_{\bar{\alpha}}^{(n)}(x, y) < \eta + \eta + d_{\bar{\alpha}}^{(0)}(x, y) < 3\eta = \varepsilon.$$ 

We obtained: for every $x, y \in X$, $\alpha \in A$, and $\bar{\alpha} \in j^{-1}(\alpha)$ the inequality $d_{\bar{\alpha}}^{(0)}(x, y) < \eta$ implies $d_{\bar{\alpha}}^{(n)}(T_n x, T_n y) < \varepsilon$ for $n > N$. On the other hand every $T_k(k = 1, \ldots, N)$ is uniformly continuous on the compact set $K$ with respect to the family $A_0$. Consequently the sequence $\{T_n\}_{n=1}^{\infty}$ is equicontinuous on $K$ with respect to $A_0$. But $K$ is compact and then pointwise convergence of $\{T_n\}_{n=1}^{\infty}$ to $T_0$ implies a uniform convergence to $T_0$ with respect to $A_0$, which completes the proof of Proposition 1.

We shall introduce the notion $j$-locally compact space. The uniform space $X$ is said to be $j$-locally compact if for every point $y_0$ and for every finite collection $\alpha_1, \ldots, \alpha_p \in A$ there exists $\varepsilon = \varepsilon(\alpha_1, \ldots, \alpha_p) > 0$ such that the set $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) = \{x \in X : d_{\alpha_i}(x, y_0) \leq \varepsilon(\alpha_1, \ldots, \alpha_p)\}$ is compact and $\varepsilon(\alpha_1, \ldots, \alpha_p) \leq \varepsilon(j(\alpha_1, \ldots, j(\alpha_p)))$ and $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon(\alpha_1, \ldots, \alpha_p)) \subset \subset K(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p)(y_0, \varepsilon(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p))$ for $\tilde{\alpha}_i \in j^{-1}(\alpha_i)$ ($i = 1, 2, \ldots, p$).

**Theorem 3.** Let $(X, A)$ be a $j$-locally compact $j$-bounded quasicomplete uniform space. The sequence $\{A_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ are as in Proposition 1. If $T_0$ is $\phi$-contractive with respect to $A_0$ and $T_n$ has a fixed point $y_n$ ($n = 0, 1, \ldots$), then the sequence $\{y_n\}_{n=1}^{\infty}$ tends to $y_0$.

**Proof.** For every finite collection $\alpha_1, \ldots, \alpha_p \in A$ we find $\varepsilon = \varepsilon(\alpha_1, \ldots, \alpha_p) > 0$ such that the set $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$ is a compact. By Proposition 1 the sequence $\{T_n\}_{n=1}^{\infty}$ tends uniformly to $T_0$ on $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)$.

Let $\tilde{\alpha}_i \in j^{-1}(\alpha_i)$ ($i = 1, 2, \ldots, p$). Then for a collection $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p$ there exists $\varepsilon = \varepsilon(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p) > 0$ such that $K(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p)(y_0, \varepsilon(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_p))$ is a compact set. Consider the continuous function $f^n_i(z) = d_{\tilde{\alpha}_i}^{(0)}(y_0, T_n z)$ on the compact set $K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon(\alpha_1, \ldots, \alpha_p))$. Having in mind the definition of $j$-locally compactness we have

$$f^n_i(z) = d_{\tilde{\alpha}_i}^{(0)}(y_0, T_n z) \leq \Phi_{\tilde{\alpha}_i}(d_{\tilde{\alpha}_i}^{(0)}(y_0, z)) \leq \Phi_{\tilde{\alpha}_i}(\varepsilon(\alpha_1, \ldots, \alpha_p)) \leq \Phi_{\tilde{\alpha}_i}(\varepsilon(j(\tilde{\alpha}_1), \ldots, j(\tilde{\alpha}_p))) \leq \varepsilon(\alpha_1, \ldots, \alpha_p)$$

for every $\tilde{\alpha}_i \in j^{-1}(\alpha_i)$. Then

$$\tilde{f}^{-i} = \sup \{f^n_i(z) : z \in K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon)\} < \varepsilon \quad (i = 1, 2, \ldots, p)$$

and $\eta = \max f^{-i} \quad i = 1, 2, \ldots, p < \varepsilon$. 

**A Continuous Dependence of Fixed Points**
In view of uniform convergence of \( \{T_n\}_{n=1}^{\infty} \) to \( T_0 \) for \( \mu = \varepsilon - \eta > 0 \) we find \( N \) such that for \( n \geq N \) and \( x \in K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) \) we have \( d_{\alpha_i}^{T_0}(T_n x, T_0 x) < \mu \). Therefore we obtain
\[
d_{\alpha_i}^{T_0}(T_n x, y_0) \leq d_{\alpha_i}^{T_0}(T_n x, T_0 x) + d_{\alpha_i}(T_0 x, T_0 y_0) \leq \mu + \varepsilon - \mu = \varepsilon,
\]
that is, for \( n \geq N \) the operator \( T_n \) maps \( K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) \) into itself.

Denote by \( T_n|K \) the restriction of \( T_n \) to \( K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) \) for each \( n \geq N \). But \( T_n|K \) is \( \phi \)-contractive with respect to \( \mathcal{A}_n \) which maps \( K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) \) into itself. On the other hand \( \mathcal{A}_0 \) and \( \mathcal{A}_n \) are equivalent and \( K(\alpha_1, \ldots, \alpha_p)(y_0, \varepsilon) \) is a compact with respect to \( \mathcal{A}_n \). Consequently \( T_n|K \) has a fixed point \( y_n \), and since \( T_n \) has only one fixed point then \( y_n \in K(\alpha_1, m \ldots, \alpha_p)(y_0, \varepsilon) \) for \( n \geq N \). It follows that \( y_n \to y_0 \).

Theorem 3 is thus proved.

\[ \square \]

**Applications**

Here we shall apply the results obtained to some initial value problems consider in [1].

Let us consider the initial value problems
\[
\varphi'(t) = F_k(t, \varphi(\Delta_1(t)), \ldots, \varphi(\Delta_m(t)), \varphi'(\tau_1(t)), \ldots, \varphi'(\tau_n(t))),
\]
(1)_k
\[
t > 0 \varphi(t) = \psi(t) , \quad \varphi'(t) = \psi'(t) , \quad t \leq 0 ,
\]
where \( \varphi(t) \) is the unknown function. The deviations \( \Delta_i(t) = \tau_i(t) \) \((i = 1, \ldots, m; i = 1, \ldots, n)\) are of mixed type and in general case unbounded. After usual transformations, assuming \( \psi(0) = 0 \) problem (1)_k can be reduced to the following one \((x(t) = \varphi'(t)) \) for \( t > 0 \) and \( \theta(t) = \psi'(t) \) for \( t \leq 0 \):

\[
x(t) = F_k(t, \int_0^{\Delta_1(t)} x(s) ds, \ldots, \int_0^{\Delta_m(t)} x(s) ds, x(\tau_1(t)), \ldots, x(\tau_n(t))),
\]
(2)_k
\[
t > 0 , \quad x(t) = \theta(t) , \quad t \leq 0 .
\]

Let \( C(R^1) \) be the linear topological space consisting of all continuous function \( f(t) : R^1 \to R^1 \) with a topology generated by a saturated family of seminorms \( \mathcal{A} = \{||\cdot||_K\} \), \( ||f||_K = \sup \{|f(t)| : t \in K\} \) where \( K \subset R^1 \) runs over all compact subsets of \( R^1 \). In view of Theorem 2 we shall look for a solution of (2)_k in a locally compact set of functions. Namely, let us consider the set \( c_L = \{f \in C(R^1) : |f(t) - f(\tilde{t})| \leq L|t - \tilde{t}| \) for every \( t, \tilde{t} \in R^1 \} \), where the Lipschitz constant \( L \) does not depend on \( K \). It easy to verify that \( c_L \) is a closed convex and every point has a neighbourhood with a compact closure by Arzela-Ascoli theorem. We shall find a solution of (2)_k in the set \( c_L^0 = \{f \in c_L : |f(t)| \leq r_0(t)\} \) where \( r_0(t) : R^1 \to R^4, \quad r_0(t) \) is continuous positive function on \( R^1 \).
We shall make the following assumption (cf. [1]):
\[
\Delta_i(t), \tau(t) : R^1_+ \to R^1 \quad (R^1_+ = [0, \infty)) \text{ are continuous,}
\]
(C1) \[
\Delta_i(0) \leq 0, \; \tau(0) \leq 0 \text{ and } |\Delta_i(t) - \Delta_i(\bar{t})| \leq P_i |t - \bar{t}|, \\
|\tau(t) - \tau(\bar{t})| \leq Q_i |t - \bar{t}|.
\]

The map \( j : A \to A \) is defined as in [1], where the index set \( A \) consists all compact subsets of \( R^1 \):
(C2) For every \( k = 0, 1, 2, \ldots \) the functions \( F_k(t, u_1, \ldots, u_m, v_1, \ldots, v_n) : R^1_+ \times R^{mn} \to R^1 \) are continuous and satisfy the conditions:
\[
|F_k(t, u_1, \ldots, u_m, v_1, \ldots, v_n)| \leq \omega(t) \quad 1 + \sum_{i=1}^{m} |u_i| + \sum_{i=1}^{n} |v_i|
\]
\[
|F_k(t, u_1, \ldots, u_m, v_1, \ldots, v_n) - F_k(t, \bar{u}_1, \ldots, \bar{u}_m, \bar{v}_1, \ldots, \bar{v}_n)| \leq \Omega [u_1 - \bar{u}_1] + \cdots + [u_m - \bar{u}_m] + [v_1 - \bar{v}_1] + \cdots + [v_n - \bar{v}_n]
\]
where \( \Omega \) is a positive constant:
\[
|F_k(t, u_1, \ldots, u_m, v_1, \ldots, v_n) - F_k(\bar{t}, u_1, \ldots, u_m, v_1, \ldots, v_n)| \leq L_0 |t - \bar{t}|
\]
where \( L_0 \) is a positive constant and
\[
L_0 + \Omega \sum_{i=1}^{m} P_i + L \sum_{i=1}^{n} Q_i \leq L ;
\]
\[
\omega(t) \quad 1 + \sum_{i=1}^{m} |\Delta_i(t)| r_0(t) + n r_0(t) \leq r_0(t); \quad \Omega (m \Delta_K + n) < 1
\]
for every compact \( K \subset R^1 \) where \( \Delta_K = \sup \{|\Delta(t)| : t \in K\} \).

Conditions (C3) and (C4) are the same as in [1], assuming that the initial functions have Lipschitz constants.

**Theorem 4.** Let the assumptions (C1) – (C4) be fulfilled. If the sequence of functions \( \{F_K\}_{K=1}^{\infty} \) tends pointwise to \( F_0 \), then the sequence of solutions of \( (2_k) \) tends to the solution of \( (2_0) \).

**Proof.** We form by the right hand side of \((2_k)\) the sequence of operators \( \{T_K\}_{K=1}^{\infty} \). It is easy to see that \( T_K \) maps the set \( C^0_1 = f \in C_L(R^1) : |f(t)| \leq r_0(t), \; t \geq 0 \) into itself. We shall verify only that \( (T_K f)(t) \) has a Lipschitz constant equals to \( L \), because another details of the proof are as in [1]. For \( t, \bar{t} > 0 \) we have
\[
|(T_K f)(t) - (T_K f)(\bar{t})| \leq L_0 |t - \bar{t}| + \Omega \sum_{i=1}^{m} r_0(t) |\Delta_i(t) - \Delta_i(\bar{t})| + \sum_{i=1}^{n} L |\tau_i(t) - \tau_i(\bar{t})| \leq L_0 |t - \bar{t}| + \Omega \sum_{i=1}^{m} r_0(t) P_i + L \sum_{i=1}^{n} Q_i |t - \bar{t}| \leq L |t - \bar{t}|.
\]
Now we can apply Theorem 2 in order to conclude that the solution of \((2_k)\) tends to the solution of \((2_0)\). This is possible because \(T_k\) is an equicontinuous family of operators and then pointwise convergence on compact sets implies a uniform convergence. \(\square\)

**References**


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