Marek Pycia
On a general solution of finite order difference equation with constant coefficients

Archivum Mathematicum, Vol. 28 (1992), No. 3-4, 237--240

Persistent URL: http://dml.cz/dmlcz/107456
ON A GENERAL SOLUTION OF FINITE ORDER
DIFFERENCE EQUATION WITH CONSTANT COEFFICIENTS

MAREK PYCIA

ABSTRACT. In the present paper we give new formulas for a general solution of the
linear difference equation of finite order with constant complex coefficients without
necessity of solving the characteristic equation.

Introduction. In this paper we deal with the following difference equation of
order $m$:

$$x_{n+m} = \sum_{r=1}^{m} a_r x_{n+m-r}$$

with constant complex coefficients $a_1, \ldots, a_m$. Our Theorem gives a simple formula
for the general solution depending only on the coefficients $a_1, \ldots, a_m$. We do not
have to solve the characteristic equation as it is usually done (cf. for instance [1],
[2]) and, in general, it is often impossible to find the exact solutions of it.

To formulate our Theorem we adopt the following convention:

$$(-1)! \cdot 0 = 1.$$

Theorem. Let $x_0, \ldots, x_{m-1}$ be arbitrary complex numbers, let $h_1, \ldots, h_m$ be
nonnegative integers. The general solution of equation (1) is of the form (2):

$$x_n = \sum_{l=0}^{m-1} \frac{(h_1 + \cdots + h_m - 1)! (h_{m-l} + \cdots + h_m)}{h_1! \cdots h_m!} \prod_{i=1}^{m} a_i^{h_i} x_{i}$$

for $n = 0, 1, \ldots$.

Proof of the Theorem. The proof is by induction with respect to $n$. 

Key words and phrases: linear difference equation, constant coefficient, general solution with-\nout characteristic equation.
Received October 14, 1991.
In the first part we show that formula (2) holds for \( n = 0, \ldots, m - 1 \). Let us take \( l \in \{0, \ldots, m - 1\} \) and consider three cases depending on \( l \) is smaller, greater than \( n \) or equal to \( n \).

In the case \( l < n \) we have \( h_1 + \cdots + mh_m = n - l > 0 \), and therefore \( h_1 + \cdots + h_m - 1 \geq 0 \). Let us note that \( h_{m-l} + \cdots + h_m = 0 \). In fact, if \( h_{m-l} + \cdots + h_m > 0 \) then would exist \( i \in \{m - l, \ldots, m\} \) such that \( h_i > 0 \). Consequently we would have \( h_1 + \cdots + mh_m \geq ih_i \geq (m - l) > n - l \); which is a contradiction. Therefore we have:

\[
\frac{(h_1 + \cdots + h_m - 1)!}{h_1! \cdots h_m!} = 0.
\]

In the case \( l > n \) we have \( h_1 + \cdots + mh_m = n - l < 0 \). Since the set of all such sequences \( (h_1, \ldots, h_m) \) is empty, the sum over this set of indices is 0.

For \( l = n \) we have \( h_1 + \cdots + mh_m = n - l = 0 \). Consequently \( h_1 = \cdots = h_m = 0 \) and, applying our convention, we get:

\[
\frac{(h_1 + \cdots + h_m - 1)!}{h_1! \cdots h_m!} = \frac{(-1)!}{1 \cdots 1} = 1.
\]

Summing up this three cases we can observe that formula (2) holds for \( n = 0, \ldots, m - 1 \).

Now, for an inductive step, we assume that Theorem is true for \( m \) consecutive indices \( n, \ldots, n + m - 1 \).

Substituting the right hand side of formula (2) into equality (1) (we change simultaneously \( n \) for \( n + m - r \) in formula (2)) and changing the order of summation we get:

\[
x_{n+m} = \sum_{r=1}^{m} \sum_{i=0}^{m-1} a_r \frac{(h_1 + \cdots + h_m - 1)!}{h_1! \cdots h_m!} a_i^h x_i = \sum_{i=1}^{m} \sum_{r=0}^{m-1} a_r \frac{(h_1 + \cdots + h_m - 1)!}{h_1! \cdots h_m!} a_i^h x_i.
\]

(3)

Let us fix \( l \in \{0, \ldots, m - 1\} \), and consider the coefficient standing before \( x_i \). Performing the indicated operations we obtain that this coefficient is equal to:

\[
c_{l_1, \ldots, l_m} a_i^{l_i}.
\]

(4)
where every $c_{g_1, \ldots, g_m}$ is uniquely defined coefficient. We will determine the value of it depending on $g_1, \ldots, g_m$. Let us fix $g_1, \ldots, g_m$.

Since we get $\sum_{i=1}^m a_i^{g_i}$ as a product of admissible $a_r$ (i.e., such that $g_r > 0$) and suitable uniquely defined $\sum_{i=1}^m a_i^{h_i}$.

$$g_i = \frac{h_i}{h_i + 1} \quad \text{for} \quad i = 1, \ldots, r - 1, r + 1, \ldots, m, \quad \text{for} \quad i = r.$$  

Therefore to obtain the coefficient $c_{g_1, \ldots, g_m}$ it is enough to add all the coefficients of the form:

$$\frac{(h_1 + \cdots + h_m - 1)! (h_{m-1} + \cdots + h_m)}{h_1! \cdots h_m!}$$

standing before admissible $\sum_{i=1}^m a_i^{h_i}$.

Let us put:

$$P(i, j) := \{ r : g_r > 0 \} \cap \{ i, \ldots, j \}.$$

We have:

$$c_{g_1, \ldots, g_m} = \frac{(h_1 + \cdots + h_m - 1)! (h_{m-1} + \cdots + h_m)}{P(1,m) h_1! \cdots h_m!} = \frac{(h_1 + \cdots + h_m - 1)! (h_{m-1} + \cdots + h_m)}{P(1,m-1) h_1! \cdots h_m!} + \frac{(h_1 + \cdots + h_m - 1)! (h_{m-1} + \cdots + h_m)}{P(m,l,m) h_1! \cdots h_m!}.$$

Because if $r \in \{ i, \ldots, j \} - P(i, j)$ then $g_r = 0$, it follows that $P(i,j) g_r = \sum_{r=i}^j g_r$. Applying formula (5), hence we get:

$$c_{g_1, \ldots, g_m} = \frac{(g_1 + \cdots + (g_r - 1) + \cdots + g_m - 1)! (g_{m-1} + \cdots + g_m)}{P(1,m-1) g_1! \cdots (g_r - 1)! \cdots g_m!} + \frac{(g_1 + \cdots + (g_r - 1) + \cdots + g_m - 1)! (g_{m-1} + \cdots + (g_r - 1) + \cdots + g_m)}{P(m,l,m) g_1! \cdots (g_r - 1)! \cdots g_m!} = \frac{g_r (g_1 + \cdots + g_m - 1)! (g_{m-1} + \cdots + g_m)}{P(1,m-1) (g_1 + \cdots + g_m - 1) \cdot g_1! \cdots g_m!} + \frac{g_r (g_1 + \cdots + g_m - 1)! (g_{m-1} + \cdots + g_m - 1)}{P(m,l,m) (g_1 + \cdots + g_m - 1) \cdot g_1! \cdots g_m!} = \sum_{r=1}^{m-1} \frac{g_r (g_1 + \cdots + g_m - 1)}{(g_1 + \cdots + g_m - 1)} + \frac{g_r (g_{m-1} + \cdots + g_m - 1)}{(g_1 + \cdots + g_m - 1) (g_{m-1} + \cdots + g_m)} \cdot \frac{g_1 + \cdots + g_m - 1)! (g_{m-1} + \cdots + g_m)}{g_1! \cdots g_m!} = \frac{g_r (g_1 + \cdots + g_m - 1)! (g_{m-1} + \cdots + g_m)}{g_1! \cdots g_m!}.$$
\[
\begin{align*}
&= \frac{(g_1 + \cdots + g_{m-i-1}) + (g_{m-i} + \cdots + g_m)(g_{m-i} + \cdots + g_m - 1)}{(g_1 + \cdots + g_m - 1)} \cdot \frac{(g_1 + \cdots + g_{m-1})(g_{m-1} + \cdots + g_m)}{g_1! \cdots g_m!} \\
&= \frac{(g_1 + \cdots + g_{m-i-1}) + (g_{m-i} + \cdots + g_m - 1)}{(g_1 + \cdots + g_m - 1)} \cdot \frac{(g_1 + \cdots + g_{m-1})(g_{m-1} + \cdots + g_m)}{g_1! \cdots g_m!} \\
&= \frac{(g_1 + \cdots + g_{m-i-1}) + (g_{m-i} + \cdots + g_m - 1)}{g_1! \cdots g_m!}.
\end{align*}
\]

Inserting this into (4) and then into (3) we obtain:

\[
x_{n+m} = \sum_{i=0}^{m-1} \frac{(g_1 + \cdots + g_{m-i-1}) + (g_{m-i} + \cdots + g_m - 1)}{(g_1 + \cdots + g_{m-1} + g_{m-i} + \cdots + g_m)} \frac{a_i^{g_i} x_{i}}{g_1! \cdots g_m!}.
\]

Now induction concludes the proof. \(\square\)

Remark 1. It may be interesting to note here that formula (2) can be written as follows:

\[
x_n = \sum_{l=0}^{m-1} \frac{(g_1 + \cdots + g_{m-i})}{g_1! \cdots g_m!} \frac{a_i^{g_i} x_{i}}{g_1! \cdots g_m!}.
\]

Remark 2. Theorem can be proved by some combinatorial reasoning.

Acknowledgement. I would like to express my thanks to Professor Janusz Matkowsi for suggestion the problem and his encouragement in writing this paper.

References