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CHARACTERIZATION OF DISTRIBUTIVE SETS BY GENERALIZED ANNIHILATORS

Radomír Halas

Abstract. Distributive ordered sets are characterized by so called generalized annihilators.

Let $L$ be a lattice. For $a, b \in L$ the annihilator $\langle a, b \rangle$ and the dual annihilator $\langle a, b \rangle_d$ of a relative to $b$ are given by $\langle a, b \rangle := \{ x \in L : x \wedge a \leq b \}$ and $\langle a, b \rangle_d := \{ x \in L : x \vee a \geq b \}$.

Several authors have studied annihilators in distributive lattices: Mandelker [1], Davey [2]; in modular lattices Davey and Nieminen [3]. In particular Mandelker proved that $L$ is distributive iff $\langle a, b \rangle$ is an ideal for all $a, b \in L$.

The aim of this paper is to characterize distributive ordered sets by the so called generalized annihilators.

Let $S$ be an ordered set, $X \subseteq S$. An upper (lower) cone of $X$ in $S$ is the set $U(X) = \{ x \in S : x \geq a \text{ for each } a \in S \}$, $(L(X) = \{ x \in S : x \geq a \text{ for each } a \in X \})$.

J. Rachůnek in [4] introduced and studied distributive and ordered sets: an ordered set $S$ is distributive if $\forall a, b, c \in S : L(U(a, b), c) = L(U(L(a, c), L(b, c)))$.

Definition 1. Let $S$ be an ordered set, $A \subseteq S$, $B \subseteq S$. A double generalized annihilator (d-annihilator) in $S$ is the set defined by

$$\langle A, B \rangle = \{ x \in S : UL(A, x) \supseteq U(B) \} , \text{ and, dually, }$$

a double generalized dual annihilator (dual d-annihilator) in $S$ is:

$$\langle A, B \rangle_d = \{ x \in S : LU(A, x) \supseteq L(B) \} .$$

If $A$ is a one element set, then the (dual) d-annihilator is called the (dual) annihilator.
Definition 2. Let $S$ be an ordered set. The subset $I \subseteq S$ is called an ideal (filter) in $S$ if it holds:

$$x, y \in I \Rightarrow LU(x, y) \subseteq I \quad (x, y \in I \Rightarrow UL(x, y) \subseteq I).$$

Remark. If $S$ is a lattice, then $I$ is an ideal (filter) in $S$ iff $I$ is a lattice ideal (filter).

Theorem 1. An ordered set $S$ is distributive if and only if each annihilator in $S$ is an ideal in $S$.

Proof. (i) Let $S$ be a distributive set, and $\langle a, B \rangle$ be an annihilator in $S$. Let $x, y \in \langle a, B \rangle$. Then $UL(a, x) \supseteq (B)$,

$$UL(a, y) \supseteq U(B).$$

Let $z \in LU(x, y)$. Then $L(z) \subseteq LU(x, y)$, $U(z) \supseteq U(x, y)$ and henceforth

$$UL(a, z) = UL(a, U(z)) \supseteq UL(a, U(x, y)).$$

By the distributive law the right side of the last inclusion is equal to

$$ULU(U(a, x), U(a, y)) = U(L(a, x), U(a, y)) = U(U(a, x) \cap U(a, y)) \supseteq U(a) \supseteq U(B),$$

hence $UL(a, z) \supseteq U(B)$, and $z \in \langle a, B \rangle$. Thus $LU(x, y) \subseteq \langle a, B \rangle$ and $\langle a, B \rangle$ is an ideal.

(ii) Let every annihilator in $S$ be an ideal, $a, b, x \in S$. Then $UL(a, x) \supseteq

$$UL(a, x) \cap UL(b, x) = U(L(a, x), L(b, x)),$$

and, analogously $UL(b, x) \supseteq U(L(a, x), L(b, x))$. Hence for $B = L(a, x) \cup L(b, x)$ it holds $a \in \langle x, B \rangle$, $b \in \langle x, B \rangle$. But $\langle x, B \rangle$ is an ideal, we have

(*)

$$LU(a, b) \subseteq \langle x, B \rangle$$

Let $z \in L(U(a, b), x)$; then $z \in LU(a, b) \cap L(x)$ and by (*) $z \in \langle x, B \rangle$. Therefore $UL(z, x) \supseteq U(L(a, x), L(b, x))$. Moreover, $x \in L(x)$ implies $L(z, x) = L(z)$, thus we obtain

$$U(z) \supseteq U(L(a, x), L(b, x)), \quad L(z) \subseteq LU(L(a, x), L(b, x)), \quad$$

i.e.

$$L(U(a, b), x) \subseteq LU(U(a, x), L(b, x)).$$

But the converse inclusion is valid for all element from $S$ (see [4]), proving distributivity of $S$. \qed

Corollary. An ordered set $S$ is distributive iff each dual annihilator in $S$ is the filter in $S$. 

Example 1. Ordered sets in Fig. 1 and Fig. 2 are not distributive (see [5]), the annihilator \( \langle a, \{c\} \rangle = \{b, c\} \) is not an ideal.

\[
\begin{array}{ccc}
\circ & \circ & \circ \\
a & b & c
\end{array}
\quad \text{Fig. 1}
\]

\[
\begin{array}{ccc}
\circ \\
d
\circ \\
\circ & \circ & \circ \\
a & b & c
\end{array}
\quad \text{Fig. 2}
\]

References


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