Włodzimierz M. Mikulski
Natural functions on $T^*T^{(r)}$ and $T^*T^{r*}$

Archivum Mathematicum, Vol. 31 (1995), No. 1, 1--7

Persistent URL: http://dml.cz/dmlcz/107519

Terms of use:

© Masaryk University, 1995

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to
digitized documents strictly for personal use. Each copy of any part of this document must contain
these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped
with digital signature within the project DML-CZ: The Czech Digital Mathematics
Library http://project.dml.cz
NATURAL FUNCTIONS ON $T^*T^{(r)}$ AND $T^*T^{r*}$

W. M. Mikulski

Abstract. We determine all natural functions on $T^*T^{(r)}$ and $T^*T^{r*}$.

All manifolds and maps are assumed to be infinitely differentiable.

1. Let $\mathcal{M}_f^n$ be the category of $n$-dimensional manifolds and their local diffeomorphisms. Consider a natural bundle $F$ over $n$-manifolds, [2].

Definition 1. A natural function $g$ on $F$ is a system of functions

$$g_M : F M \to \mathbb{R}$$

for every $n$-manifold $M$ satisfying

$$g_M = g_N \circ F f$$

for all $f : M \to N$ from $\mathcal{M}_f^n$.

Example 1. Let us remark that for every vector bundle $E \to M$, $x \in M$ and $y \in E_x$ we have a natural linear isomorphism between $E_x$ and $V_y E := T_y E_x$ given by

$$v \to \frac{d}{dt} |_{t=0} (y + tv).$$

For any vector space $W$ we have $< , > : W^* \times W \to \mathbb{R}$, $< a, v > = a(v)$.

Let $T^{(r)} = (J^r(\cdot, \mathbb{R})_0)^*$ be the linear $r$-th order tangent bundle functor and let $T^{r*} = J^r(\cdot, \mathbb{R})_0$ be the $r$-th order cotangent bundle functor, cf. [2]. For any $n$-manifold $M$ and $s \in \{1, ..., r\}$ we define $\lambda^{<s>}_M : T^*T^{(r)}M \to \mathbb{R}$ by

$$\lambda^{<s>}_M(a) := < (A^{<s>} \circ \pi)(a), q(a) >,$$

where $q : T^*T^{(r)}M \to T^{(r)}M$ is the cotangent bundle projection,

$$A^{<s>} : (T^{(r)}M)^* \to T^{r*}M \to T^{*}M \to (T^{(r)}M)^*$$

1991 Mathematics Subject Classification: 58A20, 53A55.

Key words and phrases: natural bundle, natural function.

Received June 30, 1993.
is a fibre bundle morphism over $id_M$ given by
\[ A^{<s>}((y_x^*))(x) := y_x^*(\gamma(x)), \quad \gamma : M \to \mathbb{R}, \quad \gamma(x) = 0, \quad x \in M, \]
and $\pi : T^rT^r M \to (T^r M)^*$ is a fibre bundle morphism over $id_M$ given by
\[ \pi(a) := a|V_q(a)| T^r M \cong T_x^r M, \quad a \in (T^r T^r)_x M, \quad x \in M. \]

Furthermore we define $\mu^{<s>} : T^r T^* M \to \mathbb{R}$ by
\[ \mu^{<s>} M (a) := \langle (A^{<s>} \circ q)(a), \overline{\pi}(a) \rangle, \]
where $q : T^r T^* M \to T^r M$ is the cotangent bundle projection, $A^{<s>} : T^r M \to T^r M$ is as above and $\overline{\pi} : T^r T^* M \to (T^r M)^*$ is a fibre bundle morphism over $id_M$ given by
\[ \overline{\pi}(a) := a|V_q(a)| T^r M \cong T_x^r M, \quad a \in (T^r T^r)_x M, \quad x \in M. \]

Clearly, $\{\lambda^{<s>} M \}$ is a natural function on $T^r T^r| Mf_n$ and $\{\mu^{<s>} M \}$ is a natural function on $T^r T^r| Mf_n$. 

In [1], I. Kolář has described all natural functions on $T^r F$ for $F$ from a large class of natural bundles. The method presented in [1] can not be applied in the cases $F = T^r| Mf_n$ (if $r \geq 2$) and $F = T^r| Mf_n$ because of the following reasons:
(a) If the assumptions (I), (II), (III) of [1] were satisfied for $F = T^r| Mf_n$, then using the results of [3] we could deduce that any natural function on $T^r T^r| Mf_n$ is of the form $f \circ \lambda^{<1>} M$, where $f \in C^\infty(\mathbb{R}, \mathbb{R})$. This contradicts to Theorem 1.(b)
(b) It follows from [4] that $F = T^r| Mf_n$ do not satisfy Condition (I) of [1].

In this paper we determine all natural functions on $T^r T^r| Mf_n$ and $T^r T^r| Mf_n$. We are going to prove

**Theorem 1.** All natural functions on $T^r T^r| Mf_n$ are of the form
\[ \{f \circ (\lambda^{<1>} M, \ldots, \lambda^{<r>} M) \}, \]
where $f \in C^\infty(\mathbb{R}^r)$ is a smooth function of $r$ variables.

**Theorem 2.** All natural functions on $T^r T^r| Mf_n$ are of the form
\[ \{f \circ (\mu^{<1>} M, \ldots, \mu^{<r>} M) \}, \]
where $f \in C^\infty(\mathbb{R}^r)$ is a smooth function of $r$ variables.

In the case $r = 1$ both theorems are equivalent because of a natural isomorphism $T^r T \cong T T^r$, cf. [2].

2. The proofs of Theorems 1 and 2 will be given in Item 3. In this item we prove some lemmas.

Let $q, \pi, \lambda^{<s>} M$ and $\mu^{<s>}$ be as in Example 1. The usual coordinates on $\mathbb{R}^n$ are denoted by $x^1, \ldots, x^n$ and the canonical vector fields induced by $x^1, \ldots, x^n$ on $\mathbb{R}^n$ by $\partial_1, \ldots, \partial_n$. For any vector field $X$ on $M$ the complete lift of $X$ to a natural bundle $FM$ is denoted by $FX$.

It is clear that $T^r((x^1)^*\partial_1)$ and $T^{r*}((x^1)^*\partial_1)$ are vertical over 0. We start with the proof of the following lemma.
Lemma 1. The sets
\[ \{ y \in T_0^{(r)} \mathbb{R}^n : < T^{(r)}((x^1)^r \partial_1)(y), j_0^r(x^1) > \neq 0 \} \]
and
\[ \{ y \in T_0^{(r)} \mathbb{R}^n : < T^{r*}((x^1)^r \partial_1)(j_0^r(x^1)), y > \neq 0 \} \]
are dense in \( T_0^{(r)} \mathbb{R}^n \), provided the following identifications are used:
\[ j_0^r(x^1) \in T_0^{r*} \mathbb{R}^n = (V_y T^{(r)} \mathbb{R}^n)^* \]
and
\[ (T_0^{(r)} \mathbb{R}^n)^* = V_{j_0^r(x^1)} T^{r*} \mathbb{R}^n \]
for any \( y \in T_0^{(r)} \mathbb{R}^n \).

Proof. Let \( \varphi_t \) be the flow of \((x^1)^r \partial_1 \) near \( 0 \). Then we have
\[ < T^{(r)}((x^1)^r \partial_1)(y), j_0^r(x^1) > = \left. \frac{d}{dt} \right|_{t=0} < T_0^{(r)} \varphi_t(y), j_0^r(x^1) > = \left. \frac{d}{dt} \right|_{t=0} < y, j_0^r(x^1) > = - y, \frac{d}{dt} \varphi_t^{-1} (x^1) > \]
and similarly
\[ < T^{r*}((x^1)^r \partial_1)(j_0^r(x^1)), y > = - < y, j_0^r((x^1)^r) > \]
for any \( y \in T_0^{(r)} \mathbb{R}^n \). This implies our lemma. \( \square \)

Now we prove the following lemma.

Lemma 2. Let \( g, h \) be natural functions on \( T^*T^{(r)} | M \mathcal{F}_n \) (or on \( T^*T^{r*} | M \mathcal{F}_n \)). Suppose that
\[ g_{\mathbb{R}^n}(a) = h_{\mathbb{R}^n}(a) \]
for all \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or for all \( a \in (T^*T^{r*})_0 \mathbb{R}^n \)) with
\[ \pi(a) = j_0^r(x^1) \quad (\text{or } q(a) = j_0^r(x^1)). \]
Then \( g = h \).

Proof. Consider \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or \( a \in (T^*T^{r*})_0 \mathbb{R}^n \)). Using the invariance of \( g \) and \( h \) it suffices to show that \( g_{\mathbb{R}^n}(a) = h_{\mathbb{R}^n}(a) \).
Suppose that  \( \pi(a) = j^0_0(\gamma) \) (or \( q(a) = j^0_0(\gamma) \)) for some \( \gamma : \mathbb{R}^n \to \mathbb{R} \) with \( \gamma(0) = 0 \) and \( d_0\gamma \neq 0 \). By the rank theorem there is an embedding \( \varphi : \mathbb{R}^n \to \mathbb{R}^n, \varphi(0) = 0, \) such that

\[
T^* \varphi(j^0_0(\gamma)) = j^0_0(x^1). 
\]

Then

\[
\pi(T^*T^{(r)}(\varphi(a))) = j^0_0(x^1) \quad (\text{or} \quad q(T^*T^{(r)}(\varphi(a))) = j^0_0(x^1)).
\]

Now, using the invariance of \( g \) and \( h \) with respect to \( \varphi \) and the assumption of the lemma we deduce that \( g_{R^*}(a) = h_{R^*}(a) \). Thus \( g_{R^*} = h_{R^*} \) on some dense subset in \( (T^*T^{(r)})_0 \mathbb{R}^n \) (or in \( (T^*T^{(r)})_0 \mathbb{R}^{r_n} \)). Since \( g_{R^*} \) and \( h_{R^*} \) are both of class \( C^\infty \), it holds \( g_{R^*} = h_{R^*} \) over 0.

Using Lemma 2 we prove the following lemma.

**Lemma 3.** Let \( g, h \) be natural functions on \( T^*T^{(r)}|M_{f_n} \) (or on \( T^*T^{(r)}|M_{f_n} \)). Suppose that

\[
g_{R^*}(a) = h_{R^*}(a)
\]

for all \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or for all \( a \in (T^*T^{(r)})_0 \mathbb{R}^{r_n} \) satisfying the conditions (2.1) and

\[
(2.2) \quad < a, T^{(r)}(\partial_i(q(a))) >= 0 \quad (\text{or} \quad < a, T^{(r)}(\partial_i(q(a))) > = 0)
\]

for \( i = 3, \ldots, n \). Then \( g = h \).

**Proof.** Consider \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) with \( \pi(a) = j^0_0(x^1) \) (or \( a \in (T^*T^{(r)})_0 \mathbb{R}^{r_n} \) with \( q(a) = j^0_0(x^1) \)). Using Lemma 2 it is sufficient to show that \( g_{R^*}(a) = h_{R^*}(a) \).

Define \( \Theta \in T^{(r)}_0 \mathbb{R}^n \) by

\[
< \Theta, Z(0) >= < a, T^{(r)}Z(q(a)) > \quad (\text{or} \quad < \Theta, Z(0) >= < a, T^{(r)}Z(q(a)) >)
\]

for all constant vector fields \( Z \) on \( \mathbb{R}^{r_n} \). There is a linear isomorphism \( \psi : \mathbb{R}^n \to \mathbb{R}^n \) such that \( x^1 \circ \psi = x^1 \) and

\[
T^* \psi(\Theta) = \alpha d_0 x^1 + \beta d_0 x^2
\]

for some \( \alpha, \beta \in \mathbb{R} \). Let \( \overline{a} \equiv T^*T^{(r)}(\psi(a)) \) (or \( \overline{a} \equiv T^*T^{(r)}(\psi(a)) \)). Since \( T^{(r)}(\psi(j^0_0(x^1))) = j^0_0(x^1) \), \( \overline{a} \) satisfies the condition (2.1) with \( a \) replaced by \( \overline{a} \). Moreover,

\[
< \overline{a}, T^{(r)}(\partial_i(q(\overline{a}))) >= < a, T^{(r)}((\psi^{-1})_*(\partial_i))(q(a)) >
\]

\[
= < \Theta, ((\psi^{-1})_*(\partial_i))(0) >
\]

\[
= < T^* \psi(\Theta), \partial_i(0) >= 0
\]

for \( i = 3, \ldots, n \). (Similarly,

\[
< \overline{a}, T^{(r)}(\partial_i(q(\overline{a}))) >= 0
\]

for \( i = 3, \ldots, n \). Then by the assumption of the lemma \( g_{R^*}(\overline{a}) = h_{R^*}(\overline{a}) \). Thus by the invariance of \( g \) and \( h \) with respect to \( \psi \) we obtain \( g_{R^*}(a) = h_{R^*}(a) \). **□**

Lemmas 1 and 3 imply the following assertion.
Lemma 4. Let \( g, h \) be natural functions on \( T^*T^{(r)}|\mathcal{M}f_n \) (or on \( T^*T^{**}|\mathcal{M}f_n \)). Suppose that

\[
g_{\mathbb{R}^*}(a) = h_{\mathbb{R}^*}(a)
\]

for all \( a \in (T^*T^{(r)})_0\mathbb{R}^n \) (or for all \( a \in (T^*T^{**})_0\mathbb{R}^n \)) satisfying the conditions (2.1) and (2.2) for \( i = 2, \ldots, n \). Then \( g = h \).

Proof. Consider \( a \in (T^*T^{(r)})_0\mathbb{R}^n \) (or \( a \in (T^*T^{**})_0\mathbb{R}^n \)) with (2.1) and (2.2) for \( i = 3, \ldots, n \). By Lemma 3 it suffices to show that \( g_{\mathbb{R}^*}(a) = h_{\mathbb{R}^*}(a) \).

Using the density argument and Lemma 1 we can additionally assume that

\[
< T^{(r)}((x^1)^{r}\partial_1)(q(a)), j_0^r(x^1) > = \frac{1}{\alpha}
\]

(or \( < T^{**}((x^1)^{r}\partial_1)(j_0^r(x^1)), \pi(a) > = \frac{1}{\alpha} \))

for some \( \alpha \in \mathbb{R} \).

Let \( < a, T^{(r)}\partial_2(q(a)) > = \beta \) (or \( < a, T^{**}\partial_2(q(a)) > = \beta \)). Since

\[
j_0^r-1(\partial_2 - \alpha \beta(x^1)^{r}\partial_1) = j_0^r-1(\partial_2),
\]

there exists an embedding \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n \), \( \varphi(0) = 0 \), such that:

\[
j_0^r(\varphi) = j_0^r(id),
\]

\[
ger_{0}(T\varphi \circ (\partial_2 - \alpha \beta(x^1)^{r}\partial_1)) = ger_{0}(\partial_2 \circ \varphi)
\]

and

\[
ger_{0}(T\varphi \circ \partial_1) = ger_{0}(\partial_1 \circ \varphi)
\]

for \( i = 3, \ldots, n \), cf. [2].

Let \( \overline{a} = T^*T^{(r)}\varphi(a) \) (or \( \overline{a} = T^*T^{**}\varphi(a) \)). Since \( \varphi \) preserves both \( j_0^r(x^1) \) and \( \partial_i \) for \( i = 3, \ldots, n \), then \( \overline{a} \) satisfies the conditions (2.1) and (2.2) for \( i = 3, \ldots, n \). Moreover,

\[
< \overline{a}, T^{(r)}\partial_2(q(\overline{a})) > = < a, T^*T^{(r)}\varphi^{-1}(T^{(r)}\partial_2(q(\overline{a}))) >
\]

= \( < a, T^{(r)}\partial_2(q(a)) - \alpha \beta T^{(r)}((x^1)^{r}\partial_1)(q(a)) >
\]

= \( \beta - \alpha \beta \frac{1}{\alpha} = 0
\]

(or \( < \overline{a}, T^{**}\partial_2(q(\overline{a})) > = 0 \)).

Then by the assumption of the lemma \( g_{\mathbb{R}^*}(\overline{a}) = h_{\mathbb{R}^*}(\overline{a}) \). Now, by the invariance of \( g \) and \( h \) with respect to \( \varphi \) we obtain that \( g_{\mathbb{R}^*}(a) = h_{\mathbb{R}^*}(a) \). \( \square \)

Similarly, one can prove the following assertion.
Lemma 5. Let \( g, h \) be natural functions on \( T^*T^{(r)}(M) \) (or on \( T^*T^{r*}(M) \)). Suppose that
\[
g_{\mathbb{R}^*}(a) = h_{\mathbb{R}^*}(a)
\]
for any \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or for any \( a \in (T^*T^{r*})_0 \mathbb{R}^n \)) satisfying the conditions (2.1) and (2.2) for \( i = 1, \ldots, n \). Then \( g = h \).

Proof. The proof is a replica of the proof of Lemma 4. (In the text of the proof of Lemma 4 we replace \( \partial_2 \) by \( \partial_1 \), Lemma 3 by Lemma 4 and \( i = 3, \ldots, n \) by \( i = 2, \ldots, n \).)

Now, we prove the main lemma.

Lemma 6. Let \( g, h \) be natural functions on \( T^*T^{(r)}(M) \) (or on \( T^*T^{r*}(M) \)). Suppose that
\[
g_{\mathbb{R}^*}(a) = h_{\mathbb{R}^*}(a)
\]
for every \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or for every \( a \in (T^*T^{r*})_0 \mathbb{R}^n \)) satisfying the conditions (2.1), (2.2) for \( i = 1, \ldots, n \) and
\[
< q(a), j_0^\alpha(x^\alpha) > = 0 \quad \text{(or} \quad < \overline{q}(a), j_0^\alpha(x^\alpha) > = 0 \text{)}
\]
for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \) with \( 1 \leq |\alpha| \leq r \) and \( \alpha_2 + \ldots + \alpha_n \geq 1 \). Then \( g = h \).

Proof. Consider \( a \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or \( a \in (T^*T^{r*})_0 \mathbb{R}^n \)) satisfying the conditions (2.1) and (2.2) for \( i = 1, \ldots, n \). By Lemma 5 it is sufficient to show that \( g_{\mathbb{R}^*}(a) = h_{\mathbb{R}^*}(a) \).

Let \( c_t := (x^1, tx^2, \ldots, tx^n) : \mathbb{R}^n \rightarrow \mathbb{R}^n, t \neq 0 \). It is easy to see that
\[
T^*T^{(r)}c_t(a) \rightarrow a^0 \quad \text{(or} \quad T^*T^{r*}c_t(a) \rightarrow a^0 \text{)}
\]
as \( t \rightarrow 0 \) for some \( a^0 \) satisfying (2.1), (2.2) for \( i = 1, \ldots, n \), and (2.3) for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \) with \( 1 \leq |\alpha| \leq r \) and \( \alpha_2 + \ldots + \alpha_n \geq 1 \). Then using the invariance of \( g \) and \( h \) with respect to \( c_t \) we deduce that \( g_{\mathbb{R}^*}(a) = g_{\mathbb{R}^*}(a^0) = h_{\mathbb{R}^*}(a^0) = h_{\mathbb{R}^*}(a) \).

3. We are now in position to prove both theorems. Let \( g \) be a natural function on \( T^*T^{(r)}(M) \) (or on \( T^*T^{r*}(M) \)). Define \( f : \mathbb{R}^r \rightarrow \mathbb{R} \) by
\[
f(\xi) = g_{\mathbb{R}^*}(a_\xi),
\]
where \( \xi = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r \) and \( a_\xi \in (T^*T^{(r)})_0 \mathbb{R}^n \) (or \( a_\xi \in (T^*T^{r*})_0 \mathbb{R}^n \)) is the unique form satisfying the conditions:
(2.1), (2.2) for \( i = 1, \ldots, n \), (2.3) for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n \) with \( 1 \leq |\alpha| \leq r \) and \( \alpha_2 + \ldots + \alpha_n \geq 1 \), and
\[
< q(a_\xi), j_0^\alpha((x^1)^\alpha) > = \xi_s \quad \text{(or} \quad < \overline{q}(a_\xi), j_0^\alpha((x^1)^\alpha) > = \xi_s \text{)}
\]
for $s = 1, \ldots, r$.

It is clear that $f$ is smooth. We see that

$$g_{\mathbb{R}^n}(a_\xi) = f(\lambda_{\mathbb{R}^n}^{<1>}(a_\xi), \ldots, \lambda_{\mathbb{R}^n}^{<r>}(a_\xi))$$

(or $g_{\mathbb{R}^n}(a_\xi) = f(\mu_{\mathbb{R}^n}^{<1>}(a_\xi), \ldots, \mu_{\mathbb{R}^n}^{<r>}(a_\xi))$)

for all $\xi \in \mathbb{R}^r$. Hence by Lemma 6 we obtain

$$g_M = f \circ (\lambda_{M}^{<1>}, \ldots, \lambda_{M}^{<r>})$$

(or $g_M = f \circ (\mu_{M}^{<1>}, \ldots, \mu_{M}^{<r>})$).

\Box

I would like to thank Professor I. Kolář for his comments.

References


W. M. Mikulski
Institute of Math.
Jagiellonian University
Reymonta 4, Kraków
Poland