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IMPROVEMENT OF INEQUALITIES FOR THE $(\mathbf{r},\mathbf{q})$–STRUCTURES AND SOME GEOMETRICAL CONNECTIONS

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Abstract. The main results are the inequalities (1) and (6) for the minimal number of (r,q)-structure classes,which improve the ones from [3], and also some geometrical connections, especially the inequality (13).

1. Inequalities

1.1. Definition. Let m,n,r,q be natural numbers such that $n \geq 3$ and $r \leq n$. Let M be a set which contains at least $n+q-1$ elements. Let $A = \{a_1, ..., a_n\} \subset M$. Let $P(M)$ be the set of all the subsets of M. Let the set $B = \{B_1, ..., B_m\} \subset P(M)$ fulfil the following three conditions :

(i) Each element $B_k \in B$ for $k = 1, ..., m$ contains at least r distinct elements $a_{i_1}, a_{i_2}, ..., a_{i_r} \in A$;

(ii) If $a_{i_1}, a_{i_2}, ..., a_{i_r}$ are r distinct elements of the set A, then there exist exactly q distinct elements $B_{j_1}, B_{j_2}, ..., B_{j_q} \in B$ such that for $p = 1, 2, ..., q$ we have: $a_{i_s} \in B_{j_p}$ for each $s \in \{1, 2, ..., r\}$;

(iii) For every $r+1$ distinct elements $a_{i_1}, a_{i_2}, ..., a_{i_{r+1}} \in A$ there are at most one element $B_k \in B$ such that $a_{i_s} \in B_k$ for each $s \in \{1, 2, ..., r+1\}$.

Then the ordered triplet (M, A, B) is called (r,q)-*structure.*

The elements of B are called *classes* and the elements of A are called *points*. An element $B_j \in B$ is called *class of order* k when B_j contains exactly k distinct points of A. A point $a_i \in A$ is called *point of degree* k, iff there exist exactly k distinct classes of B, which contain this point. The (r,q) -structure (M, A, B) is called *ordinary* iff all the classes of B are ordinary, i.e. contain exactly r points. We'll say that the class $B_i \in B$ is *bigger* than the class B_j iff $|B_i| > |B_j|$, and that the class $B_i \in B$ is the *biggest* iff $|B_i| \geq |B_j|$ for every $j \in \{1, 2, ..., m\}$.

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1.2. Example. Put $M = E_2$, where E_2 is the Euclidean plane.Let A be a set of n points in E_2. Let B be the set of all the straight lines determined by the points of A. Then (M, A, B) is a $(2, 1)$ -structure.

Consider the Poincaré model of the *hyperbolic* plane H_2. The points of the hyperbolic plane are interpreted as inner points of the Euclidean upper half-plane determined by the x-axis.

Horocycles (i.e. curves of identically one curvature) are either circles of the upper half-plane touching the x-axis or straight lines parallel to the x-axis. So every couple of points determines two horocycles.

1.3. Example. . Put $M = H_2$. Let A be a set of n points in H_2. Let B be the set of all the horocycles determined by the points of A. Then (M, A, B) is a $(2, 2)$ -structure.

1.4. Example. Take $M = E_2$. Let A be a set of n points in E_2 such that $\mathrm{diam}A < 2$. Let B be the set of all the unit circles containing at least two points of A. Then (M, A, B) is a $(2, 2)$ -structure.

1.5. Example. Take $M = E_2$. Let A be a set of n points in E_2, no three collinear. Let B be the set of all the circles determined by the points of A. Then (M, A, B) is a $(3, 1)$-structure.

1.6. Example. Let $M = E_3$ and $A = \{a_1, ..., a_n\} \subset M$ such that no four points from A are coplanar. Every triplet of points $a_i, a_j, a_n \in A$ uniquely determine a circle with a centre $S_{i,j,k}$. The number of such circles is finite, consequently there is a number g=the greatest of the radii. Take $G > g$ arbitrary. Now every triplet of points $a_i, a_j, a_k \in A$ determines exactly two spheres with a radius G, the centres of which are lying on the normal to that plane which is determined by the points a_i, a_j, a_k and passing through $S_{i,j,k}$. If we take just such spheres instead of B, then (M, A, B) is a $(3, 2)$-structure.

1.7. Remark. Obviously, for $q = 1$ the axiom (iii) is redundant, but for $q \geq 2$ is important and the examples 1.3 and 1.4 show its geometrical sense.

1.8. Theorem. *For any $(r, 2)$-structure, $r \geq 2$, it holds*

$$m \geq n. \tag{1}$$

Proof. For a given (r, q)-structure we denote p_k the number of points from A of degree k and t_k the number of classes from B of order k. Then

$$\sum_{k=r}^{n} kt_k = \sum_{k=q}^{m} kp_k \tag{2}$$

and

$$(3) \qquad \sum_{k=r}^{n} \binom{k}{r} t_k = q \binom{n}{r}.$$

(For the proof see [3]). Hence for $r = 2$ we get

$$m(m-1) - n(n-1) = (m-1)\sum_{k=2}^{n} t_k - \sum_{k=2}^{n} \binom{k}{2} t_k = \sum_{k=2}^{n} \{m - 1 - \binom{k}{2}\} t_k.$$

Choose $k \in \{2, 3, ..., n\}$ arbitrarily. If B contains some classes of order k, then $m \geq \binom{k}{2} + 1$, and so

$$(4) \qquad \{m - 1 - \binom{k}{2}\} t_k \geq 0.$$

If B contains no class of order k, then $t_k = 0$ and (4) holds, too. From this we get $m \geq n$ for $r = 2$. Let now $r \geq 3$ and choose a class $B^* \in B$ arbitrarily. If $|B^*| = n$ then $m \geq 1 + \binom{n}{r} \geq n$ and we have finished. Therefore, let $|B^*| \leq n - 1$. Take a point $a \in A$ such that $a \notin B^*$ and denote $B_a = \{B_i \in B; a \in B_i\}$. Now $A - a$,B_a give $(r-1, 2)$-structure and the number $m(a)$ of its classes satisfies $m(a) \leq m$. This induction argument (together with $m \geq n$ for $r = 2$) implies (1). □

1.9. Remark. The proof of inequality (1) for a (2,1)-*structure* one can find in [6].

1.10. Corollary. *The total number of horocycles determined by n points is at least n.*

1.11. Lemma. *Let (M, A, B) be a (r, q)-structure , $r \geq 2$, $q \leq r$. Let $a \in A$ and d is its degree. Then*

$$(5) \qquad d \geq \binom{D-1}{r-1}(q-1) + 1$$

where D is the order of the biggest class which contains a.

Proof. Let $a \in A$ be an arbitrary point; without loss of generality $a = a_n$. Denote by B the biggest class from B which contains a; w.l.o.g. we can fix $B = \{a_1, a_2, ..., a_{D-1}, a_n\}$. From (ii) we know that every r-tuple of points belongs exactly to q classes. So every $(r-1)$-tuple from the points $a_1, a_2, ..., a_{D-1}$ belongs - together with the point a_n - to exactly $(q-1)$ classes different from B. So we have $\binom{D-1}{r-1}(q-1)$ such classes and it is easy to see that they are distinct. □

1.12. Theorem. *For any (r, q)-structure , $r \geq 2$, $q \leq r$ it holds*

$$m \geq \sqrt{\frac{rq(q-1)}{n}\binom{n}{r}}. \tag{6}$$

Proof. Let's consider all the possible orders of classes of investigated (r, q)-structure (M, A, B) and let's write them in an increasing sequence of natural numbers $\{k_i\}_{i=0,1,\dots,s}$, precisely

$$r \leq k_0 < k_1 < k_2 < \cdots < k_s \leq n. \tag{7}$$

Let us consider now all the r-tuples of n points. They are $\binom{n}{r}$. Every r-tuple must belong to q classes. So we have to "put" in our classes $\binom{n}{r}q$ r-tuples. A class of order k_i for $i \in \{0, 1, ..., s\}$ contains $\binom{k_i}{r}$ r-tuples. From that we have

$$C = \binom{n}{r}q = \sum_{i=0}^{s} m_i \binom{k_i}{r} \leq m\binom{k_s}{r}, \tag{8}$$

where m_i is the number of classes of order k_i for $i \in \{0, 1, ..., s\}$. Further we'll denote $k = k_s$. From (8) we have

$$\binom{k}{r} \geq \frac{C}{m}. \tag{9}$$

Let $B^* \in B$ be a class of maximal order $k_s = k$. Let $a \in B^*$ an arbitrary point of this class and d the degree of this point. The order of the biggest class which contains a is of course k. Now from lemma 1.11 and inequality (9) we have

$$d \geq \binom{k-1}{r-1}(q-1) + 1 > \frac{r}{k}\binom{k}{r}(q-1) \geq \frac{rC(q-1)}{km}.$$

So

$$d \geq \frac{\binom{n}{r}rq(q-1)}{km} \geq \frac{\binom{n}{r}rq(q-1)}{nm}$$

because $k \leq n$.

Trivially, $m \geq d$ and so

$$m \geq \frac{\binom{n}{r}rq(q-1)}{nm},$$

which yields the inequality (6). □

1.13. Remark. In [3] the authors proved

$$m \geq \frac{q-1}{r-1}n. \tag{10}$$

For $r = q = 2$ the estimate (10) is better than (6), but already for $r \geq 3$ it is not so and (6) is better than (1) for $r \geq 4$, too.

1.14. Remark. In [3] the authors presented the examples of $(2, 2)$-structures for $n = 4, 7, 11$ and 16, which are showing that the estimates (1) and (10) are the best possible at least for the above-mentioned values of n.

2. Some geometrical connections

The definition of the abstract (r,q)-structure comprehends a large family of geometrical models according to certain common combinatorical features. But some concrete geometrical models are already a long time subject of interest for research. Naturally, the most investigated is the (2,1)-structure of points and straight lines in E_2 (example 1); selfevidently, in geometrical terms (see e.g.[18],[6],[10],[16],[12],[11],[5],[7]). But already the paper [6] gives the proof of inequality $m \geq n$ by means of combinatorical methods. Very natural is also the example 1.5 , i.e. the (3,1)-structure of points and circles in E_2;see [9],[2], [17]. About the (2,2)-structure from the example 4, i.e. points and unit circles it is possible to find pretty results for instance in [8], [13], [14].

Concerning the (2,2)-structure of points and horocycles Jucovič [15] asked the question, what is the minimal number $h(n)$ of horocycles determined by n points. Beck [4] proved

$$h(n) > c_5 n^2, \tag{11}$$

but his constant c_5 is extremely small. In this direction our conjecture is the following:

$$h(n) \geq \binom{n-1}{2} + 3. \tag{12}$$

Of course, it's surprising that - as we know - it has not been proved yet that every system of $n \geq 2$ points determines at least one ordinary horocycle. The reason of the troubles may be concealed in the following

2.1. Proposition. *Let A be the set of $n \geq 2$ points in a hyperbolic plane H_2. Then through every point $a_i \in A$ pass at least*

$$\frac{1+\sqrt{8n-7}}{2} \tag{13}$$

horocycles.

Proof. Consider the Poincaré model of the hyperbolic plane H_2. Choose the point $a_i \in A$ arbitrarily. Let K be any circle with centre a_i. We take the inversion with respect to K. Now the x-axis mapps into the (Euclidean) circle x' passing through a_i and every horocycle passing through a_i mapps into a straight line touching the circle x'. Let's denote the number of these touching lines by m. The intersection-points of those m lines must contain all the points of A' (perhaps with exception of a_i), therefore $\binom{m}{2} \geq n-1$. From this we obtain the asked inequality (13). □

If we take $n = \frac{j^2+j+2}{2}$, where $j = 1,2,3,...$ then $\frac{1+\sqrt{8n-7}}{2}$ is integer. In this case from the proof of the previous proposition it's obvious to construct the point-set

A of n points and to choose the point $a_i \in A$ such that through a_i are passing $\frac{1+\sqrt{8n-7}}{2} \sim \sqrt{2n}$ horocycles. So the estimate (13) is the best possible.

Compare this result with the (3,1)-structure of points and circles in E_2. There it holds (see [2]) that at least $\frac{33(n-1)}{247}$ (i.e. linearly many) ordinary circles are passing through *every* point.That is an essential difference in comparison with (13). And this is perhaps a main reason why it's so difficult to obtain a good lower estimate for the number of ordinary horocycles determined by n points.

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