Pavel Novotný
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ON PACKING OF SQUARES INTO A RECTANGLE

PAVEL NOVOTNÝ

ABSTRACT. It is proved in this paper that any system of squares with total area 1 may be packed into a rectangle whose area is less than 1.53.

The following problem is formulated in [7]: Determine the smallest number $S$ such that any system of squares with total area 1 may be (parallelly) packed into a rectangle of area $S$.

This problem was posed by L. Moser [4]. $S \geq \frac{\sqrt{2}}{2} + 1.207$ follows from considering two squares of sides $x$ and $y$, where $x > y$, $x + y = 1$ and $x(x + y)$ is maximal. Novotný [8] proved that any system of three squares with total area 1 may be packed into a rectangle of area 1.227759 (this area is necessary for packing of three squares with sides 0.7297177, 0.5588698 and 0.3939246). The four squares with sides $x = \sqrt{2} - 1$, $x = x = x = \sqrt{2}$ show that $S \geq \frac{\sqrt{2}}{2} > 1.244$.

Moon and Moser [3] found first results for the upper bound. They proved that (1) it is possible to pack any system of squares with sides $x \geq x \geq x \geq \cdots$ and with total area 1 into a square of side $a = x + \sqrt{1 - x}$.

A consequence of this is that any system of squares with total area 1 may be packed into a square of area 2.

Meir and Moser [2] extended the result (1) and they proved that (2) any system of squares with total area $V$ can be packed into a rectangle of size $a \times a$ if $a > x$, $a > x$ and $x + (a - x)(a - x) \geq V$.

Some further results for the upper bound were published by Kleitman and Krieger [1]: Any system of squares with total area 1 can be packed into a rectangle of size $\sqrt{2} \times \sqrt{2}$; its area is $\sqrt{2} \approx 1.633$. It follows from this result that (3) any system of squares with total area $V$ can be packed into a rectangle with sides $\sqrt{2V}$ and $\sqrt{V}$.

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The following theorem improves the upper estimate for $S$.

**Theorem.** Any system of squares with total area 1 may be packed into a rectangle whose area is less than 1.53.

**Proof.** We denote the squares $Q_1, Q_2, \ldots$ and their sides $x_1 \geq x_2 \geq x_3 \geq \ldots$. We shall pack the squares in the dependence upon $x_1, x_2$ as it follows:

**I.** Let $x_1 \geq \sqrt{x_2^2 - x_1^2}$. By (3) we can pack the squares $Q_1, Q_2, \ldots$ into a rectangle $P$ with sides $\sqrt{2(1 - x_1^2)}$, $\sqrt{2(1 - x_2^2)}$ and the whole system can be packed into a rectangle $R$ with sides $x_1$ and $x_2 + \sqrt{2(1 - x_1^2)}$ (Fig. 1). The area of $R$ is less than 1.53.

![Fig. 1](image1.png)

![Fig. 2](image2.png)

**II.** Let $0.645 \leq x_1 \leq \sqrt{x_2^2 - x_1^2}$. We pack the squares $Q_1, Q_2, \ldots$ as in I and all squares can be packed into a rectangle $R$ with sides $\sqrt{2(1 - x_1^2)}$ and $x_1 + \sqrt{2(1 - x_1^2)}$ (Fig. 2). The area of $R$ is less than 1.53 for every $x_1 \in (0.645, \sqrt{4/7})$.

**III.** Let $x_1 \leq 0.27$. By (1) the squares can be packed into a square $R$ of side $x_1 + \sqrt{1 - x_1^2}$; its area $1 + 2x_1 \sqrt{1 - x_1^2} < 1.53$ for every $x_1 \leq 0.27$.

It remains to investigate the domain

$$M = \{[x_1, x_2] \colon 0.27 \leq x_1 \leq 0.645, \ 0 < x_2 \leq x_1 \}.$$

**IV.** By (3) we can pack the squares $Q_1, Q_2, \ldots$ into a rectangle $P$ with sides $\sqrt{2(1 - x_1^2 - x_2^2)}$ and $\sqrt{2(1 - x_1^2 - x_2^2)}$. All squares can be packed into a rectangle $R$ by Fig. 3 if $x_1 + x_2 \geq \sqrt{2(1 - x_1^2 - x_2^2)}$, i.e. $3x_1 + 2x_2 x_3 \geq 2$, or by Fig. 4 if $x_1 + x_2 \leq \sqrt{2(1 - x_1^2 - x_2^2)}$.

The area of $R$ from Fig. 3 is

$$f(x_1, x_2) = (x_1 + x_2) \left( x_2 + \frac{2}{\sqrt{3}} \sqrt{1 - x_1 - x_2} \right).$$

We have

$$\frac{\partial f}{\partial x_1} = \frac{1}{\sqrt{3(1 - x_1 - x_2)}} \left( 2 - 4x_1 - 2x_2 - 2x_3 + (2x_1 + x_2) \sqrt{3(1 - x_1 - x_2)} \right).$$

If we denote $u(x_1, x_2) = 2 - 4x_1 - 2x_2 - 2x_3 + (2x_1 + x_2) \sqrt{3(1 - x_1 - x_2)}$, then evidently $\frac{\partial u}{\partial x_1} < 0, \frac{\partial u}{\partial x_2} < 0$ in $M$ (Fig. 11). Hence
u(x, x) ≥ u(0.645, 0.645) > 0, thus f(x, x) ≤ f(0.645, x) for [x, x] ∈ M. We verify easily that f(0.645, x) < 1.53 for every x ≤ 0.645.

\[ f(x, x) = \left( x + \frac{2}{\sqrt{3}}\sqrt{1-x-x} \right) \sqrt{2(1-x-x)}; \]

\[ \frac{\partial f}{\partial x} = \frac{\sqrt{2}}{\sqrt{1-x-x}} \left( 1-2x-x - \frac{4x}{\sqrt{3}}\sqrt{1-x-x} \right) < 0 \]

for [x, x] ∈ M (Fig. 11). Evidently \( \frac{\partial f}{\partial x} < 0 \), too, and since \( f(Z_i) < 1.53 \) for \( i \in \{1, 2, 3, 4, 5\} \), we have \( f(x, x) < 1.53 \) for every [x, x] ∈ M.

**V.** We pack the squares \( Q, Q, \ldots \) as in **IV**. All squares can be packed into a rectangle \( R \) by Fig. 5 if

\[ x + x \geq \sqrt{\frac{4(1-x-x)}{3}}, \text{ i.e. } 7x + 6x^2 + 7x \geq 4, \]

or by Fig. 6 if

\[ x + x \leq \sqrt{\frac{4(1-x-x)}{3}}. \]

The area of \( R \) from Fig. 5 is

\[ f(x, x) = (x + x) \left( x + \sqrt{2(1-x-x)} \right). \]

Since \( \frac{\partial f}{\partial x_1} > 0, \frac{\partial f}{\partial x_2} > 0 \) in \( M \) (Fig. 11) and \( f(Z_i) < 1.53 \) for \( i \in \{6, 7, 8, 9\} \), \( f(x, x) < 1.53 \) is fulfilled for every [x, x] ∈ M.

\[ \text{Fig. 5} \quad \text{Fig. 6} \]
The area of $R$ from Fig. 6 is

$$f(x, x) = \left(x + \sqrt{2(1-x-x)} \right) \frac{2}{\sqrt{3}} \sqrt{1-x-x}.$$  

Since $\frac{\partial f}{\partial x} < 0, \frac{\partial f}{\partial x} < 0$ in $M$ (Fig. 11) and $f(Z_i) < 1.53$ for $i \in \{10, 11, \ldots, 14\}$, we have $f(x, x) < 1.53$ for every $[x, x] \in M$.

VI. Let $2x \leq x$. By (2) we can pack the squares $Q, Q, \ldots$ into a rectangle $P$ with sides $a$ and $x + x$ if $x + (a-x)(x + x - x) = 1 - x - x - x (\geq 1 - x - x - x)$. It is valid for

$$a = \frac{1 - x - x - 3x + x x + x x}{x + x - x}.$$  

All squares can be packed into a rectangle $R$ by Fig. 7. Its area is

$$f(x, x, x) = (x + x)(x + a) = \frac{(x + x)(1 - x - 3x + x x + x x)}{x + x - x}.$$  

We have

$$\frac{\partial f}{\partial x} = \frac{(x + x)(1 + 2x x + 3x - 6x x - 6x x)}{(x + x - x)}.$$  

If we denote $v = 1 + 2x x + 3x - 6x x - 6x x$, then $\frac{\partial v}{\partial x} < 0$, thus $v(x, x, x) \geq v(x, x, x) = 1 - 4x x - 3x > 0$ for $[x, x] \in M$ (Fig. 11). Hence $\frac{\partial f}{\partial x} > 0$ and

$$f(x, x, x) \leq f(x, x, x) = \frac{(x + x)(1 - 3x + x x)}{x} = g(x, x).$$  

Since

$$\frac{\partial g}{\partial x} = \frac{x (x + 3x - 1)}{x} < 0 \quad \text{in } M,$$

$$g(x, x) \leq g(2x, x) = \frac{3(1 - x)}{2} < 1.5,$$

we have $f(x, x, x) \leq g(x, x) < 1.5$ for every $[x, x] \in M, x \leq x$.
VII. Let \(2x \geq x\). We pack \(Q, Q, \ldots\) as in VI. All squares can be packed into a rectangle \(R\) by Fig. 8. The area of \(R\) is

\[
f(x, x, x) = (x + x)(2x + a) = \frac{(x + x)(1 - x + x - 3x + 3x + x + 2x - x - x)}{x + x - x} = \frac{\partial f}{\partial x} = \frac{(x + x)(1 + 2x - x - 6x - 6x)}{(x + x - x)} \geq \frac{(x + x)(1 + 2x - x - 6x)}{(x + x - x)} > 0
\]

for \([x, x] \in M\) (Fig. 11), \(x \leq x\). Hence \(f(x, x, x) \leq f(x, x, x)\). Denoting

\[
h(x, x) = f(x, x, x) = \frac{(x + x)(1 - x - 3x + 3x)}{x},
\]

we have

\[
\frac{\partial h}{\partial x} = 2x - 2x + \frac{x(3x - 1)}{x} < 0, \quad \frac{\partial h}{\partial x} = 2x + \frac{-9x}{x} > 0
\]

in \(M\). It follows from this that \(h\) is maximal in \(M\) at some from the points \(Z, Z\). But \(h(Z) < 1.53, h(Z) < 1.53\).

VIII. By (2) we can pack \(Q, Q, \ldots\) into a rectangle \(P\) with sides \(x + x\) and \(a\) if \(x + (a - x)(x + 2x - x) = 1 - x - x - x\), i.e.

\[
a = \frac{1 - x - x - 3x + x + 2x x}{x + 2x - x}.
\]

All squares can be packed into a rectangle \(R\) by Fig. 9. Its area is

\[
f(x, x, x) = \frac{(x + 2x)(1 - x - 3x + 2x + 2x)}{x + 2x - x}.
\]

Since

\[
\frac{\partial f}{\partial x} = \frac{(x + 2x)(1 + 4x + 3x - 6x - 12x)}{(x + 2x - x)} \geq \frac{(x + 2x)(1 + 2x - 6x)}{(x + 2x - x)} > 0
\]

for \([x, x] \in M\) (Fig. 11), \(x \leq x\), we have \(f(x, x, x) \leq f(x, x, x)\). If we denote

\[
k(x, x) = f(x, x, x) = \frac{(x + 2x)(1 - 2x + 2x x)}{x + x},
\]

then the system

\[
\frac{\partial k}{\partial x} = \frac{x(6x + 4x x + 2x - 1)}{(x + x)} = 0,
\]
\[ \frac{\partial k}{\partial x} = \frac{x + 2x + 4x x - 10x x - 8x}{(x + x)} = 0 \]

has no solution in the interior of \( M \). Therefore the function \( k \) has a maximum on the boundary of \( M \). An easy calculation shows that this maximum is at the point \( Z \) (Fig. 11) and \( k(Z) < 1.53 \).

Further
\[
\frac{\partial f}{\partial x} = \frac{2x x + 8x x - 4x x x + 8x - 3x x - 2x x - x + 3x}{(x + 2x - x)} \geq \frac{9x - x}{(x + 2x - x)} > 0
\]

for \( [x , x ] \in M \) (Fig. 11), \( x \leq x \). It means that \( f(x , x , x ) \leq f(0.42 , x , x ) \).

If we denote \( \varphi(x , x ) = f(0.42 , x , x ) \), then

\[
\frac{\partial \varphi}{\partial x} = \frac{(0.42 + 2x)(1 + 1.68x + 3x + 3x - 2.52x - 12x x)}{0.42 + 2x - x}
\]

For \( w(x , x ) = 1 + 1.68x + 3x + 3x - 2.52x - 12x x \) and for \( x \geq 0.35 \) we have \( w(x , x ) \leq w(x , 0.35) = 3x - 2.52x + 0.4855 < 0 \) for all \( x \in (0.34, 0.39) \). Similarly, if \( x \leq 0.34 \), then \( w(x , x ) \geq w(x , 0.34) = 3x - 2.4x + 0.49 > 0 \) for \( x \in (0.34, 0.39) \). In consequence of this the function \( \varphi \) has a maximum for \( x \in (0.34, 0.35) \). We shall estimate \( \max_T \varphi(x , x ) \) for \( T = (0.34, 0.39) \times (0.34, 0.35) \). It follows from \( \frac{\partial \varphi}{\partial x} < 0, \frac{\partial w}{\partial x} < 0 \) that \(-0.041 = w(0.39, 0.35) \leq w(x , x ) \leq w(0.34, 0.34) = 0.0208 \). Since

\[
\frac{0.42 + 2x}{0.42 + 2x - x} \leq 0.42 + 0.78 < 2.2,
\]

we have in regard of (4) \( \left| \frac{\partial \varphi}{\partial x} \right| < 0.1 \) in \( T \). Further

\[
\frac{\partial \varphi}{\partial x} = (0.148176 + 1.0584x + 0.84x x - 0.84x - 8x + 14x x - 8x x + 6x - 2x )/(0.42 + 2x - x)
\]

Since the function

\[
t(x , x ) = 0.148176 + 1.0584x + 0.84x x - 0.84x - 8x + 14x x - 8x x + 6x - 2x
\]

satisfies \( \frac{\partial t}{\partial x} > 0, \frac{\partial t}{\partial x} < 0 \), we have \(-0.01885 = t(0.34, 0.35) \leq t(x , x ) \leq t(0.39, 0.34) = -0.19828 \) and because of \( 1/(0.42 + 2x - x ) < 1.8 \) we get \( |\frac{\partial \varphi}{\partial x}| < 0.04 \) in \( T \).

If \( U \subset T \) is a square with side of length 0.01, then for \( Y , Y \in U \) the inequality

\[
|\varphi(Y ) - \varphi(Y )| < 0.0014
\]

is satisfied. Since the function \( \varphi \) gets values less than 1.527 at the points \( [x , 0.34] \) for \( x \in \{0.34, 0.35, 0.36, 0.37, 0.38\} \), (5) yields \( \varphi(x , x ) < 1.53 \) in \( T \).
Let \( [x, x] \in M = (0.42, 0.50) \times (0.29, 0.37) \), \( x + x \geq x \) . By (2) we can pack \( Q, Q, \ldots \) into a rectangle \( P \) with sides \( a \) and \( x + x + x \) if

\[
x + (a - x)(x + x + x - x) = 1 - x - x - x - x - x,
\]

i.e.

\[
a = \frac{1 - x - x - x - x - 3x + x (x + x + x)}{x + x + x - x}.
\]

All squares can be packed into a rectangle \( R \) (Fig. 10) with sides \( x + x + x \), \( x + x + a \). Its area is \( f(x, x, x, x, x) = (x + x + x)(x + x + x + x + x - x - x - x - x - x - x) \). Evidently \( \frac{\partial f}{\partial x_4} > 0 \), hence

\[
f(x, x, x, x, x) \leq f(x, x, x, x, x) = m(x, x, x, x, x) = \frac{(x + x + x)(x + 2x + x + x + x + 1 - x - x - x - x)}{x + x + x - x}.
\]

Because of \( \frac{\partial m}{\partial x_4} = [x (2x + x + 2x + x + x + x + 1 + 3x - 2x - x - x - x - x + x + x + x)]/(x + x + x - x) \leq \frac{x (2 \cdot 0.5 + 3 \cdot 0.37 + 0.37 - 1 + 3 \cdot 1 - 1.74x - 0.1 \cdot 0.71 - 0.05041)}{(x + x + x - x)} = \frac{3x - 1.74x + 0.2807x - 0.05041}{(x + x + x - x)} < 0
\]

for \( x \leq 0.37 \), \( m \) has a maximum in \( M \) for \( x = 0.42 \). Similarly,

\[
\frac{\partial m}{\partial x_2} = [(x + x + x)(x + 2x)(x + x + x + x + x + 1 - x - x - x - 3x + x + x)]/(x + x + x - x) \geq [(0.71 + x)(0.42 + 2x) \cdot 0.71 - x (0.50 \cdot 0.37 + 0.74x + 0.50x + 1 - 0.42 - x + 0.50x)]/(x + x + x - x) = \frac{0.211722 + 0.2978x - 0.32x + x}{(x + x + x - x)} > 0,
\]

IX. Let \( [x, x] \in M = (0.42, 0.50) \times (0.29, 0.37) \), \( x + x \geq x \). By (2) we can pack \( Q, Q, \ldots \) into a rectangle \( P \) with sides \( a \) and \( x + x + x \) if

\[
x + (a - x)(x + x + x - x) = 1 - x - x - x - x - x,
\]

i.e.

\[
a = \frac{1 - x - x - x - x - 3x + x (x + x + x)}{x + x + x - x}.
\]

All squares can be packed into a rectangle \( R \) (Fig. 10) with sides \( x + x + x \), \( x + x + a \). Its area is \( f(x, x, x, x, x) = (x + x + x)(x + x + x + x + x - x - x - x - x - x - x) \). Evidently \( \frac{\partial f}{\partial x_4} > 0 \), hence

\[
f(x, x, x, x, x) \leq f(x, x, x, x, x) = m(x, x, x, x, x) = \frac{(x + x + x)(x + 2x + x + x + x + 1 - x - x - x - x)}{x + x + x - x}.
\]

Because of \( \frac{\partial m}{\partial x_4} = [x (2x + x + 2x + x + x + x + 1 + 3x - 2x - x - x - x - x + x + x + x)]/(x + x + x - x) \leq \frac{x (2 \cdot 0.5 + 3 \cdot 0.37 + 0.37 - 1 + 3 \cdot 1 - 1.74x - 0.1 \cdot 0.71 - 0.05041)}{(x + x + x - x)} = \frac{3x - 1.74x + 0.2807x - 0.05041}{(x + x + x - x)} < 0
\]

for \( x \leq 0.37 \), \( m \) has a maximum in \( M \) for \( x = 0.42 \). Similarly,
and hence \( m \) has a maximum for \( x = 0.37 \).

Further, on the assumptions \( x = 0.42, x = 0.37 \), using \( x \geq x \),

\[
\frac{\partial m}{\partial x} \geq \frac{0.723956 - 0.3318x - 1.58x - 0.979x - 1.16x + x + 3x - 0.42x}{(x + x + x - x)}.
\]

Since the function \( s(x, x) = 0.723956 - 0.3318x - 1.58x - 0.979x - 1.16x + x + 3x - 0.42x \) satisfies \( \frac{\partial s}{\partial x_3} < 0, \frac{\partial s}{\partial x_6} < 0 \), we have \( s(x, x) \geq s(0.37, 0.37) > 0 \), i.e. \( \frac{\partial m}{\partial x} > 0 \) and hence \( m \) has a maximum for \( x = 0.37 \). We find easily that \( m \) is maximal if \( x = \frac{-\sqrt{3.48 - \sqrt{6.8349}}}{3} \) and that the maximal value of \( f \) in \( M \) is

\[
f(0.42, 0.37, 0.37, 3.48 - \sqrt{6.8349}) < 1.53.
\]

\[\text{Fig. 11}\]

**X.** Let \( [x', x] \in M, x + x' \leq x \). As in **VIII**, the squares can be packed into a rectangle \( R \) with area

\[
f(x, x', x) = \frac{(x + 2x')(1 - x - 3x + 2x + x + 2x)}{x + 2x - x}.
\]

Since

\[
\frac{\partial f}{\partial x} \geq \frac{2x + 4x + 3x - x}{(x + 2x - x)} > 0,
\]
ON PACKING OF SQUARES INTO A RECTANGLE

\begin{equation}
\frac{\partial f}{\partial x} \geq (x + 2x)[1 + 4x + 3x + 3(x - x) - 6x (x - x) - \\
-12x (x - x)]/(x + 2x - x) = \frac{(x + 2x)(1 - 3x + 18x - 8x x)}{(x + 2x - x)} > 0
\end{equation}

for \([x, x] \subseteq M\), \(f\) is maximal for \(x = 0.5, x = 0.5 - x\). It is easy to show that \(f(0.5, x, 0.5 - x) \leq f(0.5, 0.29, 0.21) < 1.5\) for \(x \in (0.29, 0.37)\).

Since the domains \(M, \ldots, M\) cover \(M\), the proof is completed.

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PAVEL NOVOTNÝ
DEPARTMENT OF MATHEMATICS, FPEDAS
TECHNICAL UNIVERSITY OF TRANSPORT AND COMMUNICATIONS
VEĽKÝ DIEL
010 26 ŽILINA, SLOVAKIA