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UNIFORMITY AND HOMOGENEITY OF ELASTIC RODS, SHELLS AND COSSERAT THREE-DIMENSIONAL BODIES

MARCELO EPSTEIN AND MANUEL DE LEÓN

To Ivan Kolár, on the occasion of his 60th birthday.

ABSTRACT. We present a general geometrical theory of uniform bodies which includes three-dimensional Cosserat bodies, rods and shells as particular cases. Criteria of local homogeneity are given in terms on connections.

1. Introduction

An $n$-dimensional Cosserat medium $B$ is represented by an $n$-dimensional manifold $B$ which can be embedded into $\mathbb{R}^3$, the embedded submanifold endowed at each point with a deformable linearly independent basis of 3 vectors. The mechanical response is supposed to depend on the deformations of the underlying $n$-body as well as on the gradients of the attached deformable basis.

In this paper we present a general geometrical framework for arbitrary Cosserat bodies. The geometrical picture consists of an $n$-dimensional body $B$ which is embedded into the Euclidean space $\mathbb{R}^{n+m}$. The geometry of the embedding (the configuration) allows us to construct the principal bundle of linear frames of $\mathbb{R}^{n+m}$ along the embedded submanifold. Thus, a deformation is nothing but a principal bundle isomorphism of two of these configuration bundles. The constitutive equation states that the mechanical response depends on the 1-jet of the deformation. We associate to the body a groupoid of material 1-jets in such a way that the smooth uniformity is equivalent to this groupoid being a Lie groupoid. If the Cosserat body enjoys smooth global uniformity we construct a non-holonomic parallelism and, by prolongating it by means of the material symmetry group, a non-holonomic $\tilde{G}$-structure. Its integrability (integrable prolongability, in fact) is equivalent to the local homogeneity of the body. Finally, we consider the case of rods, shells and three-dimensional Cosserat bodies as particular cases.

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Our theory is based on the original works of the Cosserat brothers [5] and is the natural extension of the theory of inhomogeneities developed by Noll and Wang [43, 50, 48, 49] (see also [37, 38, 39, 40, 45]). Our approach generalizes several previous papers on three-dimensional Cosserat bodies (including second grade materials) [46, 47, 6, 7, 14, 8, 15, 10, 11, 12, 13, 41] and shells [19, 20, 21]. A general setting for continua with microstructure [2] was developed in [16, 17]. The homogeneity conditions are obtained as integrability conditions of non-holonomic parallelisms. It seems to us that non-holonomic $G$-structures deserve a careful study in order to obtain nice homogeneity conditions for Cosserat bodies. So, the results discussed, for instance, in [32, 33, 34, 35, 36, 44, 52, 25, 29, 42, 4] should be extended to that case.

2. Bundle configurations

Let $\mathcal{B}$ be an $n$-dimensional manifold. Consider an embedding $\Phi : \mathcal{B} \to \mathbb{R}^{n+m}$ of $\mathcal{B}$ into $\mathbb{R}^{n+m}$. Thus, $\Phi(\mathcal{B})$ is an $n$-dimensional embedded submanifold in $\mathbb{R}^{n+m}$. At every point $X$ in $\Phi(\mathcal{B})$ we consider the set of linear frames $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\}$ of $\mathbb{R}^{n+m}$ at $X$ such that $\{e_1, \ldots, e_n\}$ is a basis of the tangent space $T_X(\Phi(\mathcal{B}))$. Consequently, $\{e_{n+1}, \ldots, e_{n+m}\}$ is a set of linearly independent tangent vectors in $T_X \mathbb{R}^{n+m}$ which are transverse to $\Phi(\mathcal{B})$. We denote by $\mathcal{F}\Phi$ the collection of all these bases at all the points of $\Phi(\mathcal{B})$. We define a canonical projection $\pi_\Phi : \mathcal{F}\Phi \to \Phi(\mathcal{B})$ which maps a basis at $X$ onto $X$.

**Proposition 2.1.** $\mathcal{F}\Phi$ is a principal subbundle of the restriction of the linear frame bundle $\mathcal{F}\mathbb{R}^{n+m}$ of $\mathbb{R}^{n+m}$ to $\Phi(\mathcal{B})$, and whose structural group is

\[
G_0 = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \mid A \in \text{Gl}(n, \mathbb{R}), C \in \text{Gl}(m, \mathbb{R}), B \in \mathcal{M}(m,n) \right\}
\subset \text{Gl}(n + m, \mathbb{R}),
\]

where $\mathcal{M}(m,n)$ is the real vector space of matrices of order $m \times n$. $\Phi$ will be called a configuration and $\mathcal{F}\Phi$ the bundle configuration.

Given two configurations $\Phi_1, \Phi_2 : \mathcal{B} \to \mathbb{R}^{n+m}$, we put $\kappa = \Phi_2 \circ \Phi_1^{-1}$.

**Definition 2.1.** A deformation is a principal bundle isomorphism $\kappa : \mathcal{F}\Phi_1 \to \mathcal{F}\Phi_2$ between the corresponding bundle configurations which induces the identity map on the structure groups, and it covers $\kappa$.

In other words, $\kappa$ maps a basis $\{Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_{n+m}\}$ at $X \in \Phi_1(\mathcal{B})$ such that $\{Y_1, \ldots, Y_n\}$ is a basis of $T_X(\Phi_1(\mathcal{B}))$ and $\{Y_{n+1}, \ldots, Y_{n+m}\}$ are transversal to $\Phi_1(\mathcal{B})$, into a basis $\{\tilde{Y}_1, \ldots, \tilde{Y}_n, \tilde{Y}_{n+1}, \ldots, \tilde{Y}_{n+m}\}$ at $\kappa(X)$ of the same type, that is, $\{\tilde{Y}_1, \ldots, \tilde{Y}_n\}$ is a basis of $T_{\kappa(X)}(\Phi_2(\mathcal{B}))$ and $\{\tilde{Y}_{n+1}, \ldots, \tilde{Y}_{n+m}\}$ are transversal to $\Phi_2(\mathcal{B})$. With respect to these bases, $\kappa$ is given by a tensor whose associated matrix is as follows:

\[
H = \begin{pmatrix} H_1 & 0 \\ H_2 & H_3 \end{pmatrix}.
\]
where \( H_1 \in Gl(n, \mathbb{R}), H_2 \in Gl(m, \mathbb{R}), H_3 \in \mathcal{M}(m, n). \)

We fix an embedding \( \Phi_0 : B \rightarrow \mathbb{R}^{n+m} \) once and for all, and the corresponding bundle configuration \( \mathcal{F}B_{\Phi_0} \) will be denoted by \( \mathcal{E}_0 \), for brevity. We also put \( B_0 = \Phi_0(B) \).

We assume that the elastic response depends on the 1-jet of the deformation so that the constitutive equation reads as

\[
W = W_0(j_1^X \dot{\kappa}),
\]

with respect to the reference configuration \( \Phi_0 \).

**Definition 2.2.** \( B_0 \) will be called a **deformable body**.

### 3. Uniformity and Material Symmetries

**Definition 3.1.** Given a deformable body \( B_0 \) we say that it is **uniform** if for any two points \( X \) and \( Y \) in \( B_0 \) there exists a local automorphism \( \Phi \) of principal bundles of \( \mathcal{E}_0 \) from \( X \) to \( Y \) which induces the identity map between the structure groups and such that

\[
W_0(j_1^Y \dot{\kappa} \circ j_1^X \Phi) = W_0(j_1^X \dot{\kappa}),
\]

for all 1-jet of deformation \( j_1^X \dot{\kappa} \). We will call \( j_1^X \Phi \) a **material 1-jet**.

We denote by \( \Phi \) the local diffeomorphism of \( B_0 \) covered by \( \Phi \).

**Definition 3.2.** A **material symmetry** at a point \( X \in B_0 \) is a 1-jet \( j_1^X \Phi \) of a local automorphism \( \Phi \) of principal bundles of \( \mathcal{E}_0 \) at \( X \) which induces the identity map between the structure groups and such that

\[
W_0(j_1^X \dot{\kappa} \circ j_1^X \Phi) = W_0(j_1^X \dot{\kappa}),
\]

for all 1-jet of deformation \( j_1^X \dot{\kappa} \).

The following result follows immediately from the above definitions.

**Proposition 3.1.** (1) The collection \( \Omega(B_0) \) of all material 1-jets is a groupoid over \( B_0 \) with source and target projections given by \( \alpha(j_1^X \Phi) = X \) and \( \beta(j_1^X \Phi) = \Phi(X) \), respectively.

(2) The collection \( G(X) \) of all material symmetries at a point \( X \in B_0 \) has a structure of group. In fact, \( G(X) = (\alpha, \beta)^{-1}(X, X) \), where \( (\alpha, \beta) : \Omega(B_0) \rightarrow B_0 \times B_0 \) is defined by \( (\alpha, \beta)(j_1^X \Phi) = (X, \Phi(X)) \).

**Definition 3.3.** We say that \( B_0 \) enjoys smooth uniformity if \( \Omega(B_0) \) is a Lie groupoid.

In such a case, there exist local smooth uniformities (i.e., local sections of \( (\alpha, \beta) : \Omega(B_0) \rightarrow B_0 \times B_0 \)). For the sake of simplicity we will assume, from now on, that \( B_0 \) enjoys global smooth uniformity or, in other words, the Lie groupoid \( \Omega(B_0) \) is smoothly transitive.
Proposition 3.2. Assume that $\mathcal{B}_0$ enjoys smooth uniformity and take a point $X_0 \in \mathcal{B}_0$. Then $\Omega_{X_0}(\mathcal{B}_0) = \alpha^{-1}(X_0)$ is a principal bundle over $\mathcal{B}_0$ with structure group $G(X_0)$ and canonical projection $\beta$.

Proof: It follows the same lines that in Proposition 11.8 in [18].

4. Reference crystals and non-holonomic parallelisms

Consider the principal bundle $\mathcal{E}$ over $\mathbb{R}^n$ consisting of all the linear frames $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}\}$ at all the points of $\mathbb{R}^n$ such that $\{e_1, \ldots, e_n\}$ is a linear frame of $\mathbb{R}^n$. It is not hard to see that $\mathcal{E}$ is a trivial bundle, say $\mathcal{E} = \mathbb{R}^n \times G_0$.

Consider now the set $\mathcal{F}_0$ of all 1-jets $j_1^0 \tilde{\Psi}$ of local isomorphisms of principal bundles from $\mathcal{E}$ into $\mathcal{E}_0$ with source at the origin in $\mathbb{R}^n$, such that $\tilde{\Psi}$ induces the identity map between the structure groups.

It follows that $\mathcal{F}_0$ is a principal bundle over $\mathcal{B}_0$ with canonical projection $\tilde{\pi} : \mathcal{F}_0 \longrightarrow \mathcal{B}_0$, $\tilde{\pi}(j_1^0 \tilde{\Psi}) = \Psi(0)$, where $\Psi$ is the induced local diffeomorphism between the base manifolds covered by $\tilde{\Psi}$. We also have a canonical projection $\tilde{\pi}_{1,0} : \mathcal{F}_0 \longrightarrow \mathcal{E}_0$, given by $\tilde{\pi}_{1,0}(j_1^0 \tilde{\Psi}) = \Psi(0, 1)$, where $(0, 1)$ is the distinguished element of $\mathcal{E}$, i.e., $0 \in \mathbb{R}^n$ and 1 is the identity matrix in $Gl(n+m, \mathbb{R})$. (It should be noticed that the functor $\mathcal{F}$ coincides with the one previously defined by I. Kolář [29, 31].)

The element $j_1^0 \tilde{\Psi}$ will be called a non-holonomic frame at the point $\Psi(0) \in \mathcal{B}_0$. The structure group $\tilde{G}(n, n+m)$ of $\tilde{\pi} : \mathcal{F}_0 \longrightarrow \mathcal{B}_0$, consists of the 1-jets $j_1^0 \tilde{\Psi}$ of local automorphisms of $\mathcal{E}$ which induces the identity map between the structure groups and with source and target at 0.

By a direct application of chain rule, we obtain that the structure group $\tilde{G}(n, n+m)$ may be described as follows. A generic element of $\tilde{G}(n, n+m)$ is a triple $(A, B, C)$, where

$$A \in G_0, \ B \in Gl(n, \mathbb{R}), \ C \in Lin(\mathbb{R}^n, \mathfrak{g}_0),$$

where $\mathfrak{g}_0$ is the Lie algebra of $G_0$.

We will write

$$A = (A^i_j), \ B = (B^a_b^i), \ C = (C^i_{\alpha\beta}),$$

where Latin indices run from 1 to $n+m$, Greek indices run from 1 to $n$. For simplicity, we introduce new indices $a, b, c, \ldots$ running from 1 to $m$.

We have

$$A^{a+b} = 0, \text{ if } 1 \leq a \leq n, \ 1 \leq b \leq m,$$

$$C^{b}_{\alpha\gamma} = 0, \text{ if } 1 \leq \alpha \leq n, \ 1 \leq b \leq m.$$

Proposition 4.1. The group $\tilde{G}(n, n+m)$ may be identified with the semidirect product $G_0 \times Gl(n, \mathbb{R}) \times Lin(\mathbb{R}^n, \mathfrak{g}_0)$, the multiplication group given by

$$(6) \quad (A_1, B_1, C_1)(A_2, B_2, C_2) = (A, B, C),$$
where
\[ A_i^j = (A_1)_i^k (A_2)_k^j, \]
\[ B_\alpha^\beta = (B_1)_\alpha^\gamma (B_2)_\gamma^\beta, \]
\[ C_{i_\gamma} = (A_1)_i^k (B_1)_\gamma^\beta (C_2)_k^j + (A_2)_i^k (C_1)_\gamma^j. \]

**Proof:** It follows from a direct computation using the chain rule. \( \square \)

**Remark 4.1.** It should be noted that \( \dim \mathcal{G}(n, n+m) = (n+1)[n^2+nm+m^2]+n^2 \).

**Definition 4.1.** The bundle \( \mathcal{F}\mathcal{E}_0 \) will be called the non-holonomic frame bundle of \( \mathcal{E}_0 \). A global section \( \mathcal{P} \) of \( \mathcal{F}\mathcal{E}_0 \) will be called a non-holonomic parallelism on \( \mathcal{B}_0 \). A non-holonomic frame at a point \( X_0 \in \mathcal{B}_0 \) will be called a reference crystal at that point.

Suppose now that \( \mathcal{B}_0 \) enjoys smooth uniformity, and choose a crystal reference \( \mathcal{Z}_0 = j_0^1 \Psi \) at a point \( X_0 \). Given a smooth global uniformity on \( \mathcal{B}_0 \), we can transport the reference crystal at any point \( X \) in \( \mathcal{B}_0 \) by composing the uniformity from \( X_0 \) to \( X \) with the 1-jet \( j^1_{(0,1)} \Psi \). Thus, we get a global section of the bundle \( \mathcal{F}\mathcal{E}_0 \), or, in other words, a material non-holonomic parallelism \( \mathcal{P} \) on \( \mathcal{B}_0 \).

The Lie group \( G(X_0) \) can be transported via \( \mathcal{Z}_0 \) and we obtain a Lie subgroup \( \mathcal{G} \) of \( \mathcal{G}(n, n+m) \):
\[ \mathcal{G} = \mathcal{Z}_0^{-1} \circ G(X_0) \circ \mathcal{Z}_0. \]

If we prolongate \( \mathcal{P} \) by the action of \( \mathcal{G} \) we obtain a \( \mathcal{G} \)-reduction of \( \mathcal{F}\mathcal{E}_0 \). Such a reduction will be called a non-holonomic \( \mathcal{G} \)-structure on \( \mathcal{B}_0 \).

**Remark 4.2.** A classification of the subgroups of \( \mathcal{G}(n, n+m) \) could be obtained in a similar way to that in \([6, 18]\). The details of this classification as well as the integrability conditions of the corresponding \( \mathcal{G} \)-structures are matter of a future research.

Let \( (x^\alpha) \) be a coordinate system on \( \mathcal{B}_0 \) and take local bundle coordinates \( (x^\alpha, X_i^j) \) for \( \mathcal{E}_0 \). We obtain induced coordinates \( (x^\alpha, X_i^j, Y_\alpha^\beta, Z_i^j) \) on \( \mathcal{F}\mathcal{E}_0 \). We set
\[ \mathcal{P}(x^\alpha) = (x^\alpha, \mathcal{P}^j_i, Q^\beta_\alpha, R^j_i) \]  \hspace{1cm} (7)

From (7) it follows that there are \( n+m \) linearly independent vector fields \( \{\mathcal{P}_1, \ldots, \mathcal{P}_{n+m}\} \) on \( \mathbb{R}^{n+m} \) along \( \mathcal{B}_0 \) such that the first \( n \) vector fields \( \{\mathcal{P}_1, \ldots, \mathcal{P}_n\} \) define a linear parallelism on \( \mathcal{B}_0 \). These vector fields are locally given by
\[ \mathcal{P}_i = \mathcal{P}^j_i \frac{\partial}{\partial x^j}. \]

The vector fields \( \{\mathcal{P}_{n+1}, \ldots, \mathcal{P}_{n+m}\} \) are transversal to \( \mathcal{B}_0 \).

There are also \( n \) vector fields \( \{Q_1, \ldots, Q_n\} \) yielding another linear parallelism on \( \mathcal{B}_0 \), and which come from the induced diffeomorphisms on the base manifolds. Indeed, there is an underlying “uniformity” on \( \mathcal{B}_0 \) and an induced ordinary reference crystal \( j_0^1 \Psi \) at \( X_0 \) which is transported to any arbitrary point of \( \mathcal{B}_0 \).
Moreover, there exists a connection $\Gamma$ in the principal bundle $\pi_0 : \mathcal{E}_0 \to B_0$. In fact, a non-holonomic frame at a point $X \in B_0$ just defines:

1. a linear frame of $\mathbb{R}^{n+m}$ at $X$ such that its $n$ first vectors are tangent to $B_0$ and the last $m$ vectors are transversal;
2. a linear frame of $B_0$ at $X$;
3. and, a horizontal subspace at $X$ of the principal bundle $\pi_0 : \mathcal{E}_0 \to B_0$, or, in other words, an infinitesimal piece of connection.

We introduce the following notation:

$$\mathcal{N}_1 = \mathcal{P}_{n+1}, \ldots, \mathcal{N}_m = \mathcal{P}_{n+m}.$$ 

Next, take local coordinates $(x^\alpha, x^a)$ on $\mathbb{R}^{n+m}$ such that $(x^\alpha)$ are coordinates on $B_0$ and $(x^a)$ are transversal coordinates.

Thus, we have

$$\begin{align*}
Q_\alpha &= Q_\alpha^\beta (x^\gamma) \frac{\partial}{\partial x^\beta}, \\
\mathcal{P}_\alpha &= \sum_{\beta=1}^{n} \mathcal{P}_\alpha^\beta (x^\gamma) \frac{\partial}{\partial x^\beta}, \\
\mathcal{N}_a &= \sum_{\beta=1}^{n} \mathcal{P}_a^\beta (x^\gamma) \frac{\partial}{\partial x^\beta} + \sum_{b=1}^{m} \mathcal{P}_a^b (x^\gamma) \frac{\partial}{\partial x^b},
\end{align*}$$

where, for simplicity, we have written $\mathcal{P}_{n+a}^\alpha = \mathcal{P}_a^\alpha$ and $\mathcal{P}_{n+a}^a = \mathcal{P}_a^a$.

The parallelism $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ defines a linear connection $\Gamma_1$ on $B_0$ whose Christoffel components are given by

$$(\Gamma_1)_\alpha^\beta = - (\mathcal{P}^{-1})_\beta^\sigma \frac{\partial \mathcal{P}_\alpha^\gamma}{\partial x^{\sigma}}.$$ 

That is, the covariant derivative $\nabla_1$ associated with $\Gamma_1$ is given by

$$(\nabla_1)_{\cdot \cdot \cdot}^\alpha = (\Gamma_1)_\alpha^\beta \frac{\partial}{\partial x^\beta}.$$ 

The parallelism $\{Q_1, \ldots, Q_n\}$ defines another linear connection $\Gamma_2$ on $B_0$ with Christoffel components given by

$$(\Gamma_2)_\alpha^\beta = - (Q^{-1})_\beta^\sigma \frac{\partial Q_\alpha^\gamma}{\partial x^{\sigma}}.$$ 

In other words, the covariant derivative $\nabla_2$ associated with $\Gamma_2$ is given by

$$(\nabla_2)_{\cdot \cdot \cdot}^\alpha = (\Gamma_2)_\alpha^\beta \frac{\partial}{\partial x^\beta}.$$ 

Finally, let us recall the definition of the induced connection $\tilde{\Gamma}$ in $\pi_0 : \mathcal{E}_0 \to B_0$. If $\tilde{\mathcal{P}}(X) = j_0^1 \Psi$, the horizontal subspace at $\mathcal{P}(X)$ is defined to be

$$H_{\mathcal{P}(X)} = T\varphi(T_0 \mathbb{R}^n),$$
where \( \varphi : \mathbb{R}^n \rightarrow \mathcal{B}_0 \) is given by \( \varphi(r) = \tilde{\Psi}(r, 1) \). Since the horizontal lift of \( \frac{\partial}{\partial x^\alpha} \) is

\[
(\frac{\partial}{\partial x^\alpha})^H = \frac{\partial}{\partial x^\alpha} - \Gamma^j_{k\alpha} P^k_i \frac{\partial}{\partial x^j_i}
\]

we deduce that the Christoffel components of \( \Gamma \) are the following [28, 9, 30]:

\[
\Gamma^j_{i\beta} = -R^j_{i\gamma}(\mathcal{P}^{-1})^\gamma_\beta.(Q^{-1})^\gamma_\beta.
\]

A direct computation taking into account that \( \mathcal{P}^\alpha = 0 \) and \( \mathcal{R}^\alpha_{\alpha\beta} = 0 \), shows that

\[
\begin{align*}
\Gamma^\gamma_{\alpha\beta} &= -R^\gamma_{\alpha\mu}(\mathcal{P}^{-1})^\mu_\alpha(Q^{-1})^\mu_\beta, \\
\Gamma^\gamma_{\alpha\beta} &= -R^\gamma_{\alpha\mu}(\mathcal{P}^{-1})^\mu_\alpha(Q^{-1})^\mu_\beta - R^\gamma_{\alpha\mu}(\mathcal{P}^{-1})^\mu_\alpha(Q^{-1})^\mu_\beta, \\
\Gamma^\gamma_{\alpha\beta} &= -R^\gamma_{\alpha\mu}(\mathcal{P}^{-1})^\mu_\alpha(Q^{-1})^\mu_\beta - R^\gamma_{\alpha\mu}(\mathcal{P}^{-1})^\mu_\alpha(Q^{-1})^\mu_\beta.
\end{align*}
\]

Since there exists a left action of \( G_0 \) on \( \mathbb{R}^{n+m} \) we can construct an associated vector bundle with \( \mathcal{E}_0 \) which becomes the Whitney sum \( TB_0 \oplus \mathcal{N} \), where \( \mathcal{N} \) is the normal bundle generated by the vector fields \( \{ \mathcal{N}_1, \cdots, \mathcal{N}_m \} \). The connection \( \Gamma \) induces a connection in \( TB_0 \oplus \mathcal{N} \) whose tangent component defines a linear connection \( \Gamma_3 \) with covariant derivative \( \nabla_3 \) given by

\[
(\nabla_3)_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma} = \Gamma^\gamma_{\beta\alpha} \frac{\partial}{\partial x^\gamma}.
\]

Taking into account that

\[
\mathcal{P}^\beta_\beta(\mathcal{P}^{-1})^\mu_\beta + \mathcal{P}^\beta_\mu(\mathcal{P}^{-1})^\mu_\beta = 0,
\]

and putting

\[
\begin{align*}
\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} &= \Gamma^\gamma_{\beta\alpha} \frac{\partial}{\partial x^\gamma} + \Gamma^\epsilon_{\beta\alpha} \frac{\partial}{\partial x^\epsilon}, \\
\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^d} &= \Gamma^\gamma_{d\alpha} \frac{\partial}{\partial x^\gamma} + \Gamma^\epsilon_{d\alpha} \frac{\partial}{\partial x^\epsilon},
\end{align*}
\]

we compute the covariant derivative of \( \mathcal{N}_0 \) with respect to \( \Gamma \):

\[
(9) \quad \nabla_{\frac{\partial}{\partial x^\alpha}} \mathcal{N}_b = \left( \frac{\partial}{\partial x^\alpha} \mathcal{P}^\gamma_\beta - \mathcal{R}^\gamma_{\beta\epsilon}(Q^{-1})^\epsilon_\gamma \right) \frac{\partial}{\partial x^\gamma} + \left( \frac{\partial}{\partial x^\alpha} \mathcal{P}^\gamma_\epsilon - \mathcal{R}^\gamma_{\epsilon\delta}(Q^{-1})^\epsilon_\gamma \right) \frac{\partial}{\partial x^\delta}.
\]

Next, we will introduce the notion of prolongability of non-holonomic parallelisms. As we have seen, a material non-holonomic parallelism \( \tilde{\mathcal{P}} \) induces a global field of frames \( \mathcal{P} \) along \( \mathcal{B}_0 \), a linear parallelism \( \mathcal{Q} \) on \( \mathcal{B}_0 \), and a connection on the principal bundle \( \pi_0 : \mathcal{E}_0 \rightarrow \mathcal{B}_0 \). The global section \( \mathcal{P} \) of \( \pi_0 \) gives a new flat connection \( \tilde{\Gamma} \) by defining the horizontal lift of a tangent vector \( U \in T_X \mathcal{B}_0 \) as follows:

\[
U^{\tilde{\Gamma}} = T\mathcal{P}(X)(U) \in T_{\mathcal{P}(X)} \mathcal{E}_0.
\]
Thus, we have
\[
\left( \frac{\partial}{\partial x^\alpha} \right)^\mu = \frac{\partial}{\partial x^\alpha} + \frac{\partial \mathcal{P}_j^i}{\partial x^\alpha} \frac{\partial}{\partial X^j_i}.
\]

**Definition 4.2.** We say that \( \mathcal{P} \) is a prolongation if both connections, \( \Gamma \) and \( \mathcal{\tilde{\Gamma}} \), coincide. If, moreover, \( \mathcal{Q} \) is integrable, \( \mathcal{P} \) is said to be an integrable prolongation.

The reason for the above terminology is that an integrable prolongation is a non-holonomic parallelism which is obtained from \( \mathcal{P} \) and \( \mathcal{Q} \). In fact, note that a non-holonomic frame \( j_0^1 \mathcal{\tilde{\Psi}} \) at a point \( X = \mathcal{\Psi}(0) \in B_0 \) is a linear frame of \( \mathcal{E}_0 \) at the point \( \mathcal{\tilde{\Psi}}(0) \). Thus, given a global section \( \mathcal{P} \) of \( \pi_0 : \mathcal{E}_0 \to B_0 \) and a linear parallelism \( \mathcal{Q} \) of \( B_0 \), we can construct a non-holonomic parallelism denoted by \( \mathcal{P}^1(\mathcal{Q}) \) as follows: \( \mathcal{P}^1(\mathcal{Q})(X) \) is defined to be the linear frame at \( \mathcal{P}(X) \) which consists of the tangent vectors \( \{TP(X)(Q_1), \ldots, TP(X)(Q_1)\} \), completed with a suitable family of vertical tangent vectors. Of course, \( \mathcal{P}^1(\mathcal{Q}) \) defines \( \mathcal{P} \), \( \mathcal{Q} \), and the connection \( \mathcal{\tilde{\Gamma}} \).

**Proposition 4.2.** A non-holonomic parallelism \( \mathcal{P} \) is an integrable prolongation if and only if the torsion tensor \( T_2 \) of \( \Gamma_2 \), the difference tensor \( D_{13} = \nabla_1 - \nabla_3 \), and the \( m \)-forms \( \nabla N_a \), \( 1 \leq a \leq m \), simultaneously vanish.

**Proof:** If \( T_2 = 0 \), there exist local coordinates \( (x^\alpha) \) on \( B_0 \) such that
\[
Q_\alpha^\beta = \delta_\alpha^\beta,
\]
or, equivalently,
\[
Q_\alpha = \frac{\partial}{\partial x^\alpha}.
\]

Thus, the non-holonomic parallelism \( \mathcal{P} \) can be locally written as follows:
\[
\mathcal{P}(x^\alpha) = (x^\alpha, \mathcal{P}_j^i, 1, \mathcal{R}_{i\beta}^j).
\]

Moreover, the difference tensor \( D_{13} \) also vanishes. This implies that
\[
\mathcal{R}_{\alpha\beta}^\gamma = \frac{\partial \mathcal{P}_\alpha^\gamma}{\partial x^\beta}.
\]

Now, we will use that the transversal vector fields \( N_\alpha \) are parallel, and we deduce that
\[
\mathcal{R}_{b\beta}^\gamma = \frac{\partial \mathcal{P}_b^\gamma}{\partial x^\beta}, \quad \mathcal{R}_{i\beta}^c = \frac{\partial \mathcal{P}_i^c}{\partial x^\beta}.
\]

Finally, we know that
\[
\mathcal{R}_{\alpha\beta}^\gamma = 0, \quad \mathcal{P}_\alpha^\gamma = 0.
\]

Thus, the result follows.

The converse is trivial. \( \square \)

The tensors \( T_2, D_{13} \) and \( \nabla N_\alpha \) will be called the **inhomogeneity tensors** of the given material non-holonomic parallelism \( \mathcal{P} \).
Definition 4.3. A non-holonomic $\mathcal{G}$-structure on $\mathcal{B}_0$ is said to be an integrable prolongation if around each point of $\mathcal{B}_0$ there exists a local section which is an integrable prolongation.

From Proposition 4.2 it follows the following

Proposition 4.3. A non-holonomic $\mathcal{G}$-structure on $\mathcal{B}_0$ is an integrable prolongation if and only if it admits local sections whose inhomogeneity tensors vanish.

5. Homogeneity

Definition 5.1. $\mathcal{B}$ is said to be homogeneous if there exists a uniform configuration $\Phi : \mathcal{B} \rightarrow \mathbb{R}^{n+m}$ such that:

(i) $\Phi(\mathcal{B})$ is an open subset of $\mathbb{R}^n$, where $\mathbb{R}^n$ is considered as a natural subspace of $\mathbb{R}^{n+m}$ defined by the vanishing of the coordinates $x^{n+1}, x^{n+2}$ and $x^{n+m}$. Here $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+m})$ denote the standard coordinates in $\mathbb{R}^{n+m}$;

(ii) There exists a global deformation $\kappa$ from $\mathcal{F}\mathcal{B}_\Phi$ into $\mathcal{E}$ covering a global diffeomorphism $\kappa : \Phi(\mathcal{B}) \rightarrow \mathbb{R}^n$ such that $\tilde{\mathcal{P}} = \kappa^{-1}$ defines a material non-holonomic parallelism, i.e.,

$$\tilde{\mathcal{P}}(X) = j_0^1(\kappa^{-1} \circ F\tau_\kappa(X)), \forall X \in \Phi(\mathcal{B}),$$

where $\tau_\kappa(X) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the translation on $\mathbb{R}^n$ by the vector $\kappa(X)$, and $F\tau_\kappa(X)$ is the induced mapping between frame bundles.

$\mathcal{B}$ is said to be locally homogeneous if for every point $X \in \mathcal{B}$ there exists an open neighborhood which is homogeneous.

This definition is referred to a particular chosen reference crystal. More generally, we will say that $\mathcal{B}$ is homogeneous if it is homogeneous with respect to at least one reference crystal.

We will obtain a geometrical characterization of the local homogeneity.

For the sake of simplicity, we first assume that the group of material symmetries is trivial. So, we have the following

Theorem 5.1. $\mathcal{B}$ is locally homogeneous (with respect to a chosen reference crystal) if and only if there exists a uniform configuration $\Phi$ such that the associated material non-holonomic parallelism $\tilde{\mathcal{P}}$ is an integrable prolongation.

Proof: If $\mathcal{B}$ is locally homogeneous, and $\kappa$ is as in the above definition, we obtain

$$\tilde{\mathcal{P}}(x^\alpha) = (x^\alpha, P^j_i, 1, \frac{\partial P^j_i}{\partial x^\alpha}).$$

Therefore, $\tilde{\mathcal{P}}$ is an integrable prolongation.

Assume now that the inhomogeneity tensors associated with a material non-holonomic parallelism $\tilde{\mathcal{P}}$ identically vanish. We assume that $\tilde{\mathcal{P}}$ was obtained from a configuration $\Phi : \mathcal{B} \rightarrow \mathbb{R}^{n+m}$. Then, from Proposition 4.2, it is an integrable
prolongation. This means that there exist local coordinates \((x^\alpha)\) on \(\Phi(B)\) such that

\[
\tilde{P}(x^\alpha) = (x^\alpha, P^i_j, 1, \frac{\partial P^i_j}{\partial x^\alpha}).
\]

Next, we define a principal bundle automorphism

\[
\tilde{\kappa} : \tilde{FB}_B \rightarrow \mathcal{E}
\]
as follows:

\[
\tilde{\kappa}(x^\alpha, X^j_i) = (x^\alpha, P^k_i X^j_k).
\]

\(\tilde{\kappa}\) is the required deformation.\(\square\)

To end this section, we will investigate what happens if a change of reference crystal is performed. Notice that a change of reference crystal consists of composing the material non-holonomic parallelism \(\tilde{P} = (P, Q, R)\) with an element \((A, B, C)\) in the Lie group \(\tilde{G}(n, n + m)\). The new material non-holonomic parallelism is then given by \(\tilde{P}' = (P', Q', R')\), where

\[
(P')^j_i = A^k_i P^j_k, \quad (Q')^\beta_\alpha = B^\beta_\alpha Q^\gamma_\gamma, \quad (R')^j_i = A^k_i B^\beta_\gamma R^j_k + P^j_k C^k_i.
\]

So, the new connections \(\Gamma'_1\) and \(\Gamma'_2\) coincide with the former ones, \(\Gamma_1\) and \(\Gamma_2\). This fact implies that, if the torsion tensor \(T_2\) of \(\tilde{P}\) vanishes, the same is true for \(\tilde{P}'\). Therefore, the first test in order to know if a material non-holonomic parallelism is an integrable prolongation is to check the torsion tensor \(T_2\). If \(T_2\) does not vanish, we can conclude that any \(\tilde{P}\) would be not an integrable prolongation. If \(T_2\) vanishes, but the other tensors do not so, we can try for a change of reference crystal. Consider the vector fields

\[
D_{\alpha\beta} = (\nabla_1)_{\alpha} P_{\beta} - (\nabla_3)_{\alpha} P_{\beta}, \quad D_{\alpha\beta} = \nabla_{\alpha} N_{\beta}.
\]

By the same argument that in [16], we conclude the following.

**Theorem 5.2.** \(B\) is locally homogeneous if and only if there exists a uniform configuration \(\Phi\) such that the associated material non-holonomic parallelism \(\tilde{P}\) have \(T_2 = 0\) and \(D_{\alpha\beta} = 0\).

6. **Particular cases**

6.1. **Elastic rods.** (see [1, 5])

In this case, \(n = 1, m = 2\). That is, \(B_0\) is a curve in \(\mathbb{R}^3\). Since \(n = 1\), we always have that the linear parallelism \(\{Q\}\) is integrable, so that \(T_2\) identically vanishes. Proposition 4.2 becomes as follows.

**Proposition 6.1.** \(\tilde{P}\) is an integrable prolongation if and only if the difference tensor \(D_{13} = \nabla_1 - \nabla_3\), and the 1-forms \(\nabla N_1\) and \(\nabla N_2\) simultaneously vanish.
If the group of material symmetries is continuous, we obtain a $\bar{G}$-structure on the curve $B_0$, where $\bar{G}$ is a Lie subgroup of $\bar{G}(1, 3)$.

A particular case is obtained when we consider principal bundle isomorphisms $\bar{\kappa} : \mathcal{F}(\Phi_1(B)) \to \mathcal{F}(\Phi_2(B))$ such that the tangent part is precisely given by the tangent map of the induced diffeomorphisms $\kappa : \Phi_1(B) \to \Phi_2(B)$. In this case, $\mathcal{P}_1 = \mathcal{Q}_1$, and, then, $\Gamma_1 = \Gamma_2$.

### 6.2. Elastic shells.

In this case, $n = 2$, $m = 1$. That is, $B_0$ is a surface in $\mathbb{R}^3$. Thus, the non-holonomic parallelism $\bar{\mathcal{P}}$ defines two linear parallelisms $\{\mathcal{P}_1, \mathcal{P}_2\}$ and $\{\mathcal{Q}_1, \mathcal{Q}_2\}$ on the surface $B_0$, and a normal vector field $\mathcal{N}$.

Proposition 4.2 becomes as follows.

**Proposition 6.2.** $\bar{\mathcal{P}}$ is an integrable prolongation if and only if the tensor torsion $T_2$, the difference tensor $D_{13} = \nabla_1 - \nabla_3$, and the 1-form $\nabla \mathcal{N}$ simultaneously vanish.

If the group of material symmetries is continuous, we obtain a $\bar{G}$-structure on the surface $B_0$, where $\bar{G}$ is a Lie subgroup of $\bar{G}(2, 3)$.

A particular case is obtained when we consider principal bundle isomorphisms $\bar{\kappa} : \mathcal{F}(\Phi_1(B)) \to \mathcal{F}(\Phi_2(B))$ such that the tangent part is precisely given by the tangent map of the induced diffeomorphisms $\kappa : \Phi_1(B) \to \Phi_2(B)$. In this case, $\mathcal{P}_1 = \mathcal{Q}_1$, $\alpha = 1, 2$, and, then, $\Gamma_1 = \Gamma_2$.

### 6.3. Cosserat media.

Assume that $n = 3$, $m = 0$. In this case, a bundle configuration $\bar{FB}_0$ is just the linear frame bundle $\mathcal{F}(\Phi(B))$ of $\Phi(B)$, that is, the collection of all bases at all the points of $\Phi(B)$. Thus, the Lie group $G_0$ is $Gl(n, \mathbb{R})$. A deformation is a principal bundle isomorphism $\bar{\kappa} : \mathcal{F}(\Phi_1(B)) \to \mathcal{F}(\Phi_2(B))$ covering a diffeomorphism $\kappa : \Phi_1(B) \to \Phi_2(B)$. Chosen an uniform configuration $\Phi_0 : B \to \mathbb{R}^n$, we obtain a non-holonomic parallelism $\bar{\mathcal{P}} : B_0 \to \mathcal{F}E_0$ (we follow the notations introduced in the precedent sections). It should be noted that $\bar{\mathcal{F}}E_0$ is just the so-called non-holonomic second order frame bundle of $B_0$, and, hence, $\bar{\mathcal{P}}$ is a non-holonomic second order parallelism. Thus, we have two linear parallelisms $\mathcal{P}$ and $\mathcal{Q}$, and a linear connection $\Gamma$ on $B_0$. There are no transversal vector fields, and Proposition 4.2 becomes as follows.

**Proposition 6.3.** $\bar{\mathcal{P}}$ is an integrable prolongation if and only if the torsion tensor $T_2$ of $\Gamma_2$ and the difference tensor $D_{13} = \nabla_1 - \nabla_3$ simultaneously vanish.

If the group of material symmetries is continuous, we obtain a material non-holonomic second order $\bar{G}$-structure, where $\bar{G}$ is a Lie subgroup of the second order non-holonomic group $\bar{G}(n) = \bar{G}(3, 3)$.

Particular cases are obtained if we only consider deformations such that they are the natural prolongation of the diffeomorphisms between the bases, that is, $\bar{\Phi} = \mathcal{F}\Phi$. This occurs for second grade material bodies [6, 7, 8, 10, 11, 12, 13]. In this case, $\mathcal{P}_\alpha = \mathcal{Q}_\alpha$ and, hence, $\Gamma_1 = \Gamma_2$. So, we have the following.
Proposition 6.4. The following statements are equivalent:

1. \( \tilde{P} \) is an integrable prolongation;
2. it is an integrable parallelism of second order;
3. the torsion tensor \( T_2 \) of \( \Gamma_2 \) and the difference tensor \( D_{13} = \nabla_1 - \nabla_3 \) simultaneously vanish.

References


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