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HOMOMORPHISMS AND STRONG HOMOMORPHISMS OF RELATIONAL STRUCTURES

MIROSLAV NOVOTNÝ

Dedicated to the memory of Professor Otakar Borůvka

Abstract. In this paper, we present a construction of all homomorphisms of an \(n\)-ary relational structure into another \(n\)-ary relational structure. This construction may be used if constructing continuous transformations of a totally additive closure space into another space of the same type.

In the present paper we describe, among others, a construction of all homomorphisms of an \(n\)-ary relational structure into another structure of the same type. This construction belongs to a series of constructions of homomorphisms between various structures. Construction of all strong homomorphisms of an \(n\) + 1-ary relational structure into another structure of the same type was reduced to the construction of certain homomorphisms between algebras with one operation of arity \(n\) (see [5], [6], [7]). Construction of all homomorphisms of an algebra with one operation of arity \(n\) into an algebra of the same type may be reduced to the construction of certain homomorphisms of one mono-unary algebra into another one (cf. [8]). In the present paper we reduce the construction of all homomorphisms of an \(n\)-ary relational structure into another structure of the same type to the construction of all strong homomorphisms between suitable \(n\)-ary relational structures. Thus, the construction investigated here may be successively reduced to the construction of suitable homomorphisms of a mono-unary algebra into another algebra of the same type. All homomorphisms of a mono-unary algebra into another one were found in [2], [3] which was a solution of a problem formulated by O. Borůvka about 1950.

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Key words and phrases: \(n\)-ary relational structure, homomorphism of a relational structure, strong homomorphism of a relational structure, totally additive mapping, atom-preserving mapping, totally additive closure space, continuous transformation of a totally additive closure space.
Let $A$ be a set, $n \geq 2$ an integer. By an $n$-ary relation on $A$ we mean a set $r \subseteq A^n = A \times \cdots \times A$ where $A$ appears $n$ times in the last Cartesian product. The ordered pair $(A, r)$ is said to be an $n$-ary relational structure.

Let $(A', r')$ be $n$-ary relational structures, $h$ a mapping of $A$ into $A'$ such that for any $(x_1, \ldots, x_n) \in A^n$ the condition $(x_1, \ldots, x_n) \in r$ implies $(h(x_1), \ldots, h(x_n)) \in r'$. Then the mapping $h$ is called a homomorphism of $(A, r)$ into $(A', r')$.

A mapping $h$ of $A$ into $A'$ is said to be a strong homomorphism of $(A, r)$ into $(A', r')$ if it has the following property. For any $x_1, \ldots, x_{n-1}$ in $A$ and any $x_n' \in A'$ the condition $(h(x_1), \ldots, h(x_{n-1}), x_n') \in r'$ holds if and only if there exists an element $x_n$ in $A$ such that $(x_1, \ldots, x_{n-1}, x_n) \in r$ and $h(x_n) = x_n'$.

It is easy to see that any strong homomorphism is a homomorphism. The relationship between these types of homomorphisms is described by the following theorem.

**Theorem 1.** Let $n \geq 2$ be an integer, $(A, r)$, $(A', r')$ $n$-ary relational structures, $h$ a mapping of $A$ into $A'$. Then the following assertions are equivalent.

(i) $h$ is a homomorphism of $(A, r)$ into $(A', r')$.

(ii) There exist an $n$-ary relation $\overline{r} \supseteq r$ on $A$ and an $n$-ary relation $\overline{r}' \subseteq r'$ on $A'$ such that $h$ is a strong homomorphism of $(A, \overline{r})$ into $(A', \overline{r}')$.

**Proof.** Let (i) hold. Put $\overline{r}' = \{(h(x_1), \ldots, h(x_n)) \in (A')^n; (x_1, \ldots, x_n) \in r, \overline{r} = \{(x_1, \ldots, x_n) \in A^n; (h(x_1), \ldots, h(x_n)) \in \overline{r}'\}$.

If $(x'_1, \ldots, x'_n) \in \overline{r}'$, there exist $x_1, \ldots, x_n$ in $A$ such that $(x_1, \ldots, x_n) \in r$ and $h(x_i) = x'_i$ for any $i$ with $1 \leq i \leq n$. It follows that $(x'_1, \ldots, x'_n) = (h(x_1), \ldots, h(x_n)) \in r'$. Thus, $\overline{r}' \subseteq r'$.

If $(x_1, \ldots, x_n) \in r$, then $(h(x_1), \ldots, h(x_n)) \in \overline{r}'$ which implies that $(x_1, \ldots, x_n) \in \overline{r}$. Hence $r \subseteq \overline{r}$.

Suppose that $x_1, \ldots, x_{n-1}$ in $A$, $x'_n \in A'$ are arbitrary.

(a) If $(h(x_1), \ldots, h(x_{n-1}), x'_n) \in \overline{r}'$, there exist $y_1, \ldots, y_n$ in $A$ such that $(y_1, \ldots, y_n) \in r$ and $h(y_1) = h(x_1), \ldots, h(y_{n-1}) = h(x_{n-1})$, $h(y_n) = x'_n$. Since $(h(x_1), \ldots, h(x_{n-1}), h(y_n)) \in \overline{r}'$, we obtain $(x_1, \ldots, x_{n-1}, y_n) \in \overline{r}$.

(b) If there exists $x_n$ in $A$ such that $(x_1, \ldots, x_{n-1}, x_n) \in \overline{r}$ and $h(x_n) = x'_n$, we have $(h(x_1), \ldots, h(x_{n-1}), x'_n) = (h(x_1), \ldots, h(x_{n-1}), h(x_n)) \in \overline{r}'$.

Thus, $h$ is a strong homomorphism of $(A, \overline{r})$ into $(A', \overline{r}')$ and (ii) holds.

Let (ii) hold. Suppose that $x_1, \ldots, x_n$ in $A$ are arbitrary. If $(x_1, \ldots, x_n) \in r$, then $(x_1, \ldots, x_n) \in \overline{r}$. Put $x'_n = h(x_n)$. Since $h$ is a strong homomorphism of $(A, \overline{r})$ into $(A', \overline{r}')$, we obtain $(h(x_1), \ldots, h(x_{n-1}), x'_n) \in \overline{r}' \subseteq r'$. Thus, $h$ is a homomorphism of $(A, r)$ into $(A', r')$ and (i) holds. $\square$

As a consequence we obtain

**Construction of all homomorphisms**

Let $n \geq 2$ be an integer, let $n$-ary relational structures $(A, r)$, $(A', r')$ be given. Choose an $n$-ary relation $\overline{r} \supseteq r$ on $A$ and an $n$-ary relation $\overline{r}' \subseteq r'$ on $A'$.

Construct all strong homomorphisms of $(A, \overline{r})$ into $(A', \overline{r}')$ using $[7]$. 

Any of them is a homomorphism of \((A, r)\) into \((A', r')\) and any homomorphism of \((A, r)\) into \((A', r')\) may be constructed in this way by a suitable choice of \(\overline{r}\) and \(\overline{r}'\).

In examples 1 and 2 we meet the construction of all strong homomorphisms of a binary relational structure \((A, t)\) into a structure \((A', t')\) of the same type. By [5] we construct \((P(A), P[t])\) where \(P(A)\) is the power set of \(A\) and \(P[t](X) = \{y \in A; \text{there exists } x \in X \text{ with } (x, y) \in t\}\) for any \(X \in P(A)\). Clearly, \((P(A), P[t])\) is a mono-unary algebra. Similarly, we construct the mono-unary algebra \((P(A'), P[t'])\). The construction of all strong homomorphisms of \((A, t)\) into \((A', t')\) means to construct all totally additive and atom-preserving homomorphisms of \((P(A), P[t])\) into \((P(A'), P[t'])\).

A mapping \(H\) of \(P(A)\) into \(P(A')\) is called totally additive if \(H(\bigcup_{i \in I} X_i) = \bigcup_{i \in I} H(X_i)\) for any system of sets \((X_i)_{i \in I}\) where \(X_i \in P(A)\) for any \(i \in I\). A mapping \(H\) of \(P(A)\) into \(P(A')\) is referred to as atom-preserving if for any \(x \in A\) there exists \(x' \in A'\) such that \(H(\{x\}) = \{x'\}\).

Thus, we construct all homomorphisms of \((P(A), P[t])\) into \((P(A'), P[t'])\) according to [2] and [3] and reject all of them that are not totally additive and atom-preserving. If \(H\) is a totally additive atom-preserving homomorphism of \((P(A), P[t])\) into \((P(A'), P[t'])\), then we put \(h(x) = H(\{x\})\) for any \(x \in A\). The mapping \(h\) is a strong homomorphism of \((A, t)\) into \((A', t')\) and any strong homomorphism of \((A, t)\) into \((A', t')\) may be constructed in this way. For the details see [5].

**Example 1.** Let us have two binary relational structures \((A, r)\), \((A', r')\) where 
\[ A = \{a, b, c\}, \quad A' = \{a', b', c'\} \]
and the relations \(r, r'\) are given by the following tables.

<table>
<thead>
<tr>
<th>(r)</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(b)</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>(c)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(r')</th>
<th>(a')</th>
<th>(b')</th>
<th>(c')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a')</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(b')</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(c')</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We now define the relations \(\overline{r}\), \(\overline{r}'\) by the following tables.

<table>
<thead>
<tr>
<th>(\overline{r})</th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(b)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(c)</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\overline{r}')</th>
<th>(a')</th>
<th>(b')</th>
<th>(c')</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a')</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(b')</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(c')</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

We construct the mono-unary algebras \((P(A), P[\overline{r}])\),  \((P(A'), P[\overline{r}'])\) (see Fig. 1). For the operations \(P[\overline{r}]\), \(P[\overline{r}']\) we obtain the following tables.

\[
\begin{array}{c|cccccccc}
X & \emptyset & \{a\} & \{b\} & \{c\} & \{a, b\} & \{a, c\} & \{b, c\} & \{a, b, c\} \\
P[\overline{r}](X) & \emptyset & \{a, b, c\} & \{b, c\} & \{a, b, c\} & \{a, c\} & \{b, c\} & \{a, b, c\}
\end{array}
\]

\[
\begin{array}{c|cccccccc}
X & \emptyset & \{a'\} & \{b'\} & \{c'\} & \{a', b'\} & \{a', c'\} & \{b', c'\} & \{a', b', c'\} \\
P[\overline{r}'](X) & \emptyset & \{a', c'\} & \emptyset & \{a', c'\} & \{b', c'\} & \{a', c'\} & \{c'\} & \{a', c'\}
\end{array}
\]
Let us choose an arbitrary totally additive atom-preserving homomorphism $H$ of the mono-unary algebra $(P(A), P[\tau])$ into $(P(A'), P[\tau'])$. It is easy to see that we may take $H(\{a, b, c\}) = \{a', c'\}$, $H(\{b, c\}) = \{c'\}$ because $H$ assigns elements of cycles in the second algebra to elements of cycles in the first algebra. Since $H$ is atom-preserving, we obtain $H(\{b\}) = \{c'\}$, $H(\{c\}) = \{c'\}$, $H(\{a\}) = \{a'\}$. It is easy to see that $H$ may be extended to a totally additive atom-preserving homomorphism of $(P(A), P[\tau])$ into $(P(A'), P[\tau'])$. Putting $H(\{x\}) = h(x)$ for any $x \in A$ we obtain a strong homomorphism of the structure $(A, \tau)$ into $(A', \tau')$ which is a homomorphism of the structure $(A, \tau)$ into $(A', \tau')$. We have $h(a) = a'$, $h(b) = c' = h(c)$.

**Example 2.** Let two binary relational structures $(A, r)$, $(A', r')$ be given where $A = \{a, b\}$, $A' = \{a', b'\}$ and the relations $r$, $r'$ are defined by the following tables.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$r'$</th>
<th>$a'$</th>
<th>$b'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a'$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$b'$</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

![Diagram](image)

$$(P(A), P[\tau]) \quad (P(A'), P[\tau'])$$

Fig. 1

We put $\tau = r$, $\tau' = r'$. Then the operations $P[\tau]$, $P[\tau']$ have the following tables (see Fig. 2).
An atom-preserving homomorphism $H$ of $(P(A), P[\bar{r}])$ into $(P(A'), P[\bar{r}'])$ assigns to any atom $\{x\} \in P(A), x \in A$ an atom $\{x'\} \in P(A'), x' \in A'$. Since the elements $\{a\}, \{b\}$ form a cycle, the elements $H(\{a\}), H(\{b\})$ form a cycle, too. Clearly $H(\{a\}) = \emptyset = H(\{b\})$ contradicts the hypothesis that $H$ is atom-preserving. For the same reason the case $H(\{a\}) = \{a', b'\} = H(\{b\})$ is impossible. Thus we have $H(\{a\}) = \{b'\} = H(\{b\}), H(\emptyset) = \emptyset, H(\{a,b\}) = \{b'\}$. Then $H$ is a totally additive atom-preserving homomorphism of $(P(A), P[\bar{r}])$ into $(P(A'), P[\bar{r}'])$. It follows that the mapping $h$ defined by $h(a) = b' = h(b)$ is a strong homomorphism of $(A, \bar{r})$ into $(A', \bar{r}')$, i.e., a homomorphism of $(A, r)$ into $(A', r')$.

**Example 3.** Let $A \neq \emptyset$ be a set, $r$ a ternary relation on $A$ such that $(x, y, z) \in r$ implies $x = y = z$. Let $A' \neq \emptyset$ be a set and $r'$ a ternary relation on $A'$ with
the following property: If \( x' \in A' \), \( y' \in A' \), then \((x', y', y') \in r'\). In particular, \((x', x', x') \in r'\) holds for any \( x' \in A'\).

We put \( \mathcal{F} = \{(x, x, x); x \in A\} \), \( \mathcal{F}' = \{(x', x', x'); x' \in A'\} \). Then, clearly, \( r \subseteq \mathcal{F}, \mathcal{F}' \subseteq r' \). It is easy to see that a mapping \( h \) of \( A \) into \( A' \) is a strong homomorphism of \((A, \mathcal{F})\) into \((A', \mathcal{F}')\) if and only if it is injective; for the details see Example 3 of [1]. It follows that any injective mapping of \( A \) into \( A' \) is a homomorphism of \((A, \mathcal{F})\) into \((A', \mathcal{F}')\).

On the other hand, it is easy to see that any mapping \( h \) of \( A \) into \( A' \) is a homomorphism of the structure \((A, \mathcal{r})\) into \((A', \mathcal{r}')\). The corresponding ternary relations \( \mathcal{F} \) and \( \mathcal{F}' \) are described in the proof of Theorem 1. We have \( \mathcal{F}' = \{(h(x), h(x), h(x)); (x, x, x) \in \mathcal{r}\}, \mathcal{F} = \{(x, y, z); (h(x), h(y), h(z)) \in \mathcal{r}'\} \). Clearly, \( h \) is a strong homomorphism of \((A, \mathcal{F})\) into \((A', \mathcal{F}')\).

**Example 4.** Let \( m \geq 1 \), \( n \geq 1 \) be integers such that \( n \) divides \( m \), suppose \( A = \{a_1, \ldots, a_m\} \), \( A' = \{a'_1, \ldots, a'_n\} \) where \( a_i \neq a_j \) for any \( i, j \) with \( 1 \leq i < j \leq m \) and \( a'_i \neq a'_j \) for any \( i, j \) satisfying \( 1 \leq i < j \leq n \). Let us have \( f(a_i) = a_{i+1} \) for any \( i \) with \( 1 \leq i < m \), \( f(a_m) = a_1 \), \( f'(a'_i) = a'_{i+1} \) for any \( i \) with \( 1 \leq i < n \), \( f'(a'_n) = a'_1 \). Then \((A, f), (A', f')\) are mono-unary algebras that can be regarded as binary relational structures. If putting \( h(a_i) = a'_i \) where \( i \equiv j \mod n \) we obtain a homomorphism of the algebra \((A, f)\) onto \((A', f')\) that may be regarded as a strong homomorphism of the binary relational structure \((A, f)\) onto \((A', f')\). If choosing an arbitrary binary relation \( r \) on \( A \) such that \( r \subseteq f \) and an arbitrary binary relation \( r' \) on \( A' \) such that \( f' \subseteq r' \), then \( h \) is a homomorphism of \((A, \mathcal{r})\) onto \((A', \mathcal{r}')\).

The presented examples are intended to demonstrate our Construction in a transparent way. For this reason the sets appearing in Example 1 and 2 have small cardinalities. Besides, these examples present the construction of one homomorphism using one possible choice of \( \mathcal{F} \) and \( \mathcal{F}' \); the remaining cases can be solved in a similar way. Naturally, all homomorphisms of a relational structure \((A, \mathcal{r})\) into \((A', \mathcal{r}')\) may be constructed simply by testing all mappings of \( A \) into \( A' \) and by rejecting all that are not homomorphisms. Our Construction offers another way for solving this problem. Example 4 presents another application of Theorem 1: construction of some pairs of binary relational structures with a prescribed homomorphism.

Construction of all strong homomorphisms of one \( n \)-ary relational structure into another one that appears as a step in our Construction is transformed in construction of all homomorphisms of one mono-\( n - 1 \)-ary algebra into another algebra of the same type in [5], [6], [7]. By [8] the construction of all homomorphisms of one mono-\( n - 1 \)-ary algebra into another one may be reduced to construction of all so called decomposable homomorphisms of one mono-unary algebra of a particular class into another one (cf. [2], [3], [4]). These constructions demonstrate the fundamental meaning of mono-unary algebras in some problems concerning homomorphisms of algebraic structures.
We now present an application of homomorphisms between relational structures.

Let \( A \) be a set. A mapping \( R \) of \( P(A) \) into \( P(A) \) is said to be a closure on \( A \) if it is extensive, monotone, and idempotent. An ordered pair \((P(A), R)\) is referred to as a closure space if \( R \) is a closure on \( A \). This closure space is called totally additive if so is \( R \). By Lemma 4, Theorem 3 and Theorem 4 of [5], \( R \) is a totally additive closure on \( A \) if and only if \( R = P[r] \) holds for some preordering \( r \) on \( A \). This preordering \( r \) may be obtained from \( R \) by means of an operator \( Q \) that is defined as follows.

Let \( A \) be a set, \( R \) a mapping of \( P(A) \) into itself. We put \( Q[R] = \{(x, y) \in A \times A; y \in R(\{x\})\} \). By Theorem 3 of [5], \( Q[R] \) is a preordering for any closure space \((A, R)\). By Lemma 4 of [5], \( R = P[Q[R]] \) holds if and only if \( R \) is totally additive.

Let \((P(A), R), (P(A'), R')\) be totally additive closure spaces. A mapping \( h \) of \( A \) into \( A' \) is called a continuous transformation of \((P(A), R)\) into \((P(A'), R')\) if \( P[h](R(X)) \subseteq R'(P[h](X)) \) holds for any \( X \in P(A) \). A mapping \( h \) of \( A \) into \( A' \) is referred to as a continuous and closed transformation of \((P(A), R)\) into \((P(A'), R')\) if \( P[h](R(X)) = R'(P[h](X)) \) is satisfied for any \( X \in P(A) \). By Theorem 6 of [5] a mapping \( h \) of \( A \) into \( A' \) is a continuous and closed transformation of \((P(A), R)\) into \((P(A'), R')\) if and only if it is a strong homomorphism of \((A, Q[R])\) into \((A', Q[R'])\).

**Theorem 2.** Let \((P(A), R), (P(A'), R')\) be totally additive closure spaces, \( h \) a mapping of \( A \) into \( A' \). Then the following assertions are equivalent.

(i) \( h \) is a continuous transformation of \((P(A), R)\) into \((P(A'), R')\).

(ii) \( h \) is a homomorphism of \((A, Q[R])\) into \((A', Q[R'])\).

**Proof.** Put \( r = Q[R], r' = Q[R'] \).

Let (i) hold and suppose that \((x, y) \in r\) which implies that \( h(y) \in P[h](P[r](\{x\})) \) or \( P[h](P[Q[R]](\{x\})) = P[h](R(\{x\})) \subseteq R'(P[h](\{x\})) \) or \( P[Q[R']](P[h](\{x\})) = P[r'](\{h(x)\}) \) and, therefore, \((h(x), h(y)) \in r'\). Hence (ii) holds.

Let (ii) hold and suppose that \( y' \in P[h](R(X)) \) where \( X \in P(A) \) is arbitrary. Since \( R = P[r] \), there exists \( y \in P[r](X) \) such that \( h(y) = y' \). Thus there exists \( x \in X \) such that \((x, y) \in r\) which implies \((h(x), h(y)) \in r'\). Since \( h(x) \in P[h](X) \), we obtain \( y' = h(y) \in P[r'](P[h](X)) \). Thus, \( P[h](R(X)) = P[h](P[r](X)) \subseteq P[r'](P[h](X)) = R'(P[h](X)) \) holds for any \( X \in P(A) \) and (i) is satisfied. \( \square \)

It follows that constructions of continuous transformations of \((P(A), R)\) into \((P(A'), R')\) may be reduced to constructions of homomorphisms of \((A, Q[R])\) into \((A', Q[R'])\) where \( Q[R], Q[R'] \) are preorderings.
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