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Róbert Vrábeľ

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QUASILINEAR AND QUADRATIC SINGULARLY PERTURBED PERIODIC BOUNDARY VALUE PROBLEM

RÓBERT VRÁBEL'

Abstract. The problem of existence and asymptotic behavior of solutions of the quasilinear and quadratic singularly perturbed periodic boundary value problem as a small parameter at highest derivative tends to zero is studied.

1. Introduction

In the paper [3] the author established the sufficient conditions for existence and uniform convergence of the solutions of semilinear singularly perturbed differential equation $\epsilon y'' = f(t,y)$ to a solution of the reduced problem $f(t,u) = 0$ as the small positive parameter ϵ tends to zero. The purpose of this paper is an extension of lemma 1 above cited paper on the more general cases. We will consider periodic boundary value problem problem

$$\text{(PBVP}_0) \qquad \begin{aligned} \epsilon y'' &= F\left(t, y, y'\right), \qquad a < t < b\,, \\ y(a,\epsilon) - y(b,\epsilon) &= 0, \quad y'(a,\epsilon) - y'(b,\epsilon) = 0\,, \end{aligned}$$

where $F \in C^1\left([a,b] \times \mathbb{R}^2\right)$ and ϵ is a small positive parameter. The proofs of the theorems are based upon the method of lower and upper solutions.

As usual, we say that $\alpha \in C^2([a,b])$ is a lower solution for (PBVP$_0$) if $\alpha(a,\epsilon) - \alpha(b,\epsilon) = 0$, $\alpha'(a,\epsilon) - \alpha'(b,\epsilon) \geq 0$, and $\epsilon\alpha''(t,\epsilon) \geq F\left(t, \alpha(t,\epsilon), \alpha'(t,\epsilon)\right)$ for every $t \in [a,b]$. An upper solution $\beta \in C^2([a,b])$ satisfies $\beta(a,\epsilon) - \beta(b,\epsilon) = 0$, $\beta'(a,\epsilon) - \beta'(b,\epsilon) \leq 0$, and $\epsilon\beta''(t,\epsilon) \leq F\left(t, \beta(t,\epsilon), \beta'(t,\epsilon)\right)$ for every $t \in [a,b]$.

Definition ([2])**.** We say that the function F satisfies a Bernstein-Nagumo condition if for each $M > 0$ there exists a continuous function $h_M : [0,\infty) \to [a_M, \infty)$ with $a_M > 0$ and $\int_0^\infty \frac{s}{h_M(s)} \mathrm{d}s = \infty$ such that for all $y, |y| \leq M$, all $t \in [a,b]$ and all $z \in \mathbb{R}$

$$|F(t,y,z)| \leq h_M(|z|)\,.$$

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Remark 1. As a remark we conclude that the functions of the form $F(t, y, y') = f(t, y)y' + g(t, y)$ and $F(t, y, y') = f(t, y)y'^2 + g(t, y)$ satisfy Bernstein-Nagumo condition.

Lemma ([1],[2]). *If α, β are lower and upper solutions for* (PBVP_0) *such that $\alpha(t, \epsilon) \le \beta(t, \epsilon)$ on $[a, b]$ and F satisfies a Bernstein-Nagumo condition, then there exists a solution y of* (PBVP_0) *with $\alpha(t, \epsilon) \le y(t, \epsilon) \le \beta(t, \epsilon), a \le t \le b$.*

Notation. Let

$$\begin{aligned}
D_\delta(u) &= \left\{(t, y) \in \mathbb{R}^2 : a \le t \le b, |y - u(t)| < d(t)\right\},\\
D_{\delta,a}(u) &= \left\{(t, y) \in \mathbb{R}^2 : a \le t \le a + \tfrac{\delta}{2}, y \in \mathbb{R}\right\} \cap D_\delta(u),\\
D_{\delta,b}(u) &= \left\{(t, y) \in \mathbb{R}^2 : b - \tfrac{\delta}{2} \le t \le b, y \in \mathbb{R}\right\} \cap D_\delta(u),\\
D^+_{\delta,a}(u) &= \left\{(t, y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \ge 0\right\} \cap D_{\delta,a}(u),\\
D^-_{\delta,a}(u) &= \left\{(t, y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \le 0\right\} \cap D_{\delta,a}(u),\\
D^+_{\delta,b}(u) &= \left\{(t, y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \ge 0\right\} \cap D_{\delta,b}(u),\\
D^-_{\delta,b}(u) &= \left\{(t, y) \in \mathbb{R}^2 : a \le t \le b, y - u(t) \le 0\right\} \cap D_{\delta,b}(u),
\end{aligned}$$

where $d(t)$ is the positive continuous function on $[a, b]$ such that

$$d(t) = |u(a) - u(b)| + \delta \qquad \text{for} \quad a \le t \le a + \frac{\delta}{2} \text{ and } \; b - \frac{\delta}{2} \le t \le b$$

and

$$d(t) = \delta \qquad \text{for } a + \delta \le t \le b - \delta\,,$$

δ is a small positive constant and $u = u(t)$ is a solution of the reduced problem $F(t, u, u') = 0$ defined on $[a, b]$ such that $u \in C^2([a, b])$. Next, we use notation $h(t, y)$ for $F(t, y, u'(t))$.

2. Quasilinear periodic boundary value problem

In this section we consider the quasilinear periodic boundary value problem

$$(\mathrm{PBVP}_1) \qquad \begin{aligned} \epsilon y'' &= f(t, y)y' + g(t, y), \qquad a < t < b\,,\\ y(a, \epsilon) - y(b, \epsilon) &= 0, \quad y'(a, \epsilon) - y'(b, \epsilon) = 0\,, \end{aligned}$$

where $f, g \in C^1(D_\delta(u))$. About the behavior of solutions of (PBVP_1) for $\epsilon \to 0^+$ states the following result.

Theorem 1. *Consider the problem* (PBVP_1). *Let δ, m be the positive constants such that $\frac{\partial h(t,y)}{\partial y} \ge m$ for every $(t, y) \in D_\delta(u)$. Let $f(t, y) \le 0$ and $f(t, y) \ge 0$ for every $(t, y) \in D_{\delta,a}(u)$ and $(t, y) \in D_{\delta,b}(u)$, respectively. Then there exists ϵ_0 such that for any $\epsilon \in (0, \epsilon_0]$ the problem* (PBVP_1) *has solution satisfying the inequality*

$$-v_{11} - v_2 - C\epsilon \le y(t, \epsilon) - u(t) \le v_{12} + v_2 + C\epsilon$$

for $u(a) \ge u(b)$ and

$$v_{12} - v_2 - C\epsilon \le y(t, \epsilon) - u(t) \le -v_{11} + v_2 + C\epsilon$$

for $u(a) \le u(b)$, where

$$v_{11}(t,\epsilon) = (u(a) - u(b)) \frac{\exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(t-b)\right]}{\exp\left[\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-a)\right] - 1}$$

$$v_{12}(t,\epsilon) = (u(a) - u(b)) \frac{\exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(a-t)\right]}{\exp\left[\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-a)\right] - 1}$$

$$v_2(t,\epsilon) = |u'(a) - u'(b)| \frac{\exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(t-a)\right] + \exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-t)\right]}{2\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}\left(1 - \exp\left[-\left(\frac{m}{\epsilon}\right)^{\frac{1}{2}}(b-a)\right]\right)},$$

and C is a positive constant.

Proof. For $u(a) \ge u(b)$ we define the lower solutions

$$\alpha(t,\epsilon) = u(t) - v_{11}(t,\epsilon) - v_2(t,\epsilon) - \Gamma(\epsilon)$$

and the upper solutions by

$$\beta(t,\epsilon) = u(t) + v_{12}(t,\epsilon) + v_2(t,\epsilon) + \Gamma(\epsilon)$$

(in the case $u(a) \le u(b)$ we proceed analogously).

Here $\Gamma(\epsilon) = \frac{\epsilon\gamma}{m}$, where γ is a constant which will be defined below. One can easily check that the function α, β satisfy the boundary conditions required for the lower and upper solutions of (PBVP_1) and $\alpha \le \beta$ on $[a,b]$. Now we show that $\epsilon\alpha''(t,\epsilon) \ge f(t,\alpha(t,\epsilon))\alpha'(t,\epsilon) + g(t,\alpha(t,\epsilon))$ and $\epsilon\beta''(t,\epsilon) \le f(t,\beta(t,\epsilon))\beta'(t,\epsilon) + g(t,\beta(t,\epsilon))$ on $[a,b]$. By Taylor theorem we obtain

$$\begin{aligned}
\epsilon\alpha'' - F(t,\alpha,\alpha') &= \epsilon\alpha'' - (F(t,\alpha,\alpha') - F(t,u,u')) \\
&= \epsilon\alpha'' - [(F(t,\alpha,u') - F(t,u,u')) + (F(t,\alpha,\alpha') - F(t,\alpha,u'))] \\
&= \epsilon\alpha'' - \left[\frac{\partial h(t,\eta(t,\epsilon))}{\partial y}(\alpha - u) + f(t,\alpha)(\alpha' - u')\right] \\
&= \epsilon u'' - \epsilon v_{11}'' - \epsilon v_2'' + \frac{\partial h(t,\eta)}{\partial y}(v_{11} + v_2 + \Gamma) + f(t,\alpha)(v_{11}' + v_2') \\
&\ge \epsilon u'' - \epsilon v_{11}'' - \epsilon v_2'' + m(v_{11} + v_2 + \Gamma) + f(t,\alpha)(v_{11}' + v_2') \\
&= \epsilon u'' + \epsilon\gamma + f(t,\alpha)(v_{11}' + v_2') \\
&\ge -\epsilon|u''| + \epsilon\gamma + f(t,\alpha)(v_{11}' + v_2')
\end{aligned}$$

and

$$F(t,\beta,\beta') - \epsilon\beta'' \ge -\epsilon|u''| + \epsilon\gamma + f(t,\beta)(v_{12}' + v_2'),$$

where $(t,\eta(t,\epsilon))$ is a point between $(t,\alpha(t,\epsilon))$ and $(t,u(t))$, $(t,\eta(t,\epsilon)) \in D_\delta(u)$ for sufficiently small ϵ.

Let $|u'(a) - u'(b)| > 0$. From the above assumptions we obtain that

$$f(t,\alpha)\,(v'_{11} + v'_2) \geq 0 \text{ and } \; f(t,\beta)\,(v'_{12} + v'_2) \geq 0$$

on $\left[a, a + \frac{\tilde{\delta}}{2}\right] \cup \left[b - \frac{\tilde{\delta}}{2}, b\right]$ for $\epsilon \in (0, \epsilon_1]$ where $\tilde{\delta} = \min\{\delta, \delta_1\}$, and δ_1, ϵ_1 are such that $v'_{11} + v'_2 < 0$ $(v'_{12} + v'_2 < 0)$ $(v'_{11} + v'_2 > 0$ $(v'_{12} + v'_2 > 0)$ on $[a, a + \delta_1]$ $([b - \delta_1, b])$ and $(t, \alpha) \subset D_{\tilde{\delta}}(u)$, $(t, \beta) \subset D_{\tilde{\delta}}(u)$ for $\epsilon \in (0, \epsilon_1]$. On the interval $\left[a + \frac{\tilde{\delta}}{2}, b - \frac{\tilde{\delta}}{2}\right]$ is $|f(t,\alpha)\,(v'_{11} + v'_2)| \leq c_1\epsilon$ and $|f(t,\beta)\,(v'_{12} + v'_2)| \leq c_1\epsilon$ for sufficiently small ϵ, for instance, when $\epsilon \in (0, \epsilon_0]$, $\epsilon_0 \leq \epsilon_1$ and suitable positive constant c_1.

If $|u'(a) - u'(b)| = 0$ then $v'_{11} \leq 0$ $(v'_{12} \geq 0)$ and $|f(t,\alpha)v'_{11}| \leq c_1\epsilon$ and $|f(t,\beta)v'_{12}| \leq c_1\epsilon$ on $\left[a + \frac{\tilde{\delta}}{2}, b\right]$ $\left(\left[a, b - \frac{\tilde{\delta}}{2}\right]\right)$.

Thus if we choose a constant $\gamma \geq c_1 + \max\{|u''(t)|, t \in [a,b]\}$ then $\epsilon\alpha''(t,\epsilon) \geq f(t,\alpha(t,\epsilon))\alpha'(t,\epsilon) + g(t,\alpha(t,\epsilon))$ and $\epsilon\beta''(t,\epsilon) \leq f(t,\beta(t,\epsilon))\beta'(t,\epsilon) + g(t,\beta(t,\epsilon))$ on $[a,b]$. The existence of solution of (PBVP_1) satisfying the stated inequalities follows from lemma. This completes the proof. □

Example. As an illustrative example we consider the (PBVP_1) for differential equation $\epsilon y'' = yy' - \left(t - \frac{1}{2}\right)$ on $[0,1]$. General solution of the reduced problem $uu' - \left(t - \frac{1}{2}\right) = 0$ is $u^2 = t^2 - t + k, k \in \mathbb{R}$; however, only $u(t) = t - \frac{1}{2}$ satisfies the assumptions asked on the solution of reduced problem. On the basis of theorem 1, there is ϵ_0 such that for every $\epsilon \in (0, \epsilon_0]$ has the problem solution satisfying

$$v_{12} - v_2 - C\epsilon \leq y(t,\epsilon) - \left(t - \frac{1}{2}\right) \leq -v_{11} + v_2 + C\epsilon$$

on $[0,1]$.

3. Quadratic periodic boundary value problem

Now we will consider the quadratic periodic boundary value problem

$$(\mathrm{PBVP}_2) \qquad \begin{aligned} \epsilon y'' &= f(t,y)y'^2 + g(t,y), \qquad a < t < b\,, \\ y(a,\epsilon) - y(b,\epsilon) &= 0, \quad y'(a,\epsilon) - y'(b,\epsilon) = 0\,, \end{aligned}$$

where $f, g \in C^1(D_\delta(u))$.

In the following theorem we use notation $f(t,y)$ arb M when we do not impose any restrictions on the sign of the function f in the set $M \subset \mathbb{E}_2$.

Theorem 2. *Consider the problem* (PBVP_2). *Let* δ, m *be the positive constants such that* $\frac{\partial h(t,y)}{\partial y} \geq m$ *for every* $(t,y) \in D_\delta(u)$. *Let* $u(a) \geq u(b)$ *and*

1. $f(t,y) \leq 0, (t,y) \in D_{\delta,a}(u)$ $(f(t,y) \geq 0, (t,y) \in D_{\delta,b}(u))$ *for* $4u'(a) > |u'(a) - u'(b)| > 0$ $(4u'(b) > |u'(a) - u'(b)| > 0)$.
2. $f(t,y) \equiv 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$ $(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,b}(u))$ *for* $|u'(a) - u'(b)| > 4u'(a) > 0$ $(|u'(a) - u'(b)| > 4u'(b) > 0)$.

3_1. $f(t,y) \equiv 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) > 0, 4u'(t) < |u'(a) - u'(b)|, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) > 0, 4u'(t) < |u'(a) - u'(b)|, t \in (b-\omega, b), \omega > 0)$.

3_2. $f(t,y) \leq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \geq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) > 0, 4u'(t) > |u'(a) - u'(b)|, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) > 0, 4u'(t) > |u'(a) - u'(b)|, t \in (b-\omega, b), \omega > 0)$.

3_3. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) > 0, 4u'(t) \equiv |u'(a) - u'(b)|, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) > 0, 4u'(t) \equiv |u'(a) - u'(b)|, t \in (b-\omega, b), \omega > 0)$.

4_1. $f(t,y) \equiv 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 4u'(a) = 0, 4u'(t) > 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| > 4u'(b) = 0, 4u'(t) > 0, t \in (b-\omega, b), \omega > 0)$.

4_2. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 4u'(a) = 0, 4u'(t) < 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| > 4u'(b) = 0, 4u'(t) < 0, t \in (b-\omega, b), \omega > 0)$.

4_3. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 4u'(a) = 0, 4u'(t) \equiv 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| > 4u'(b) = 0, 4u'(t) \equiv 0, t \in (b-\omega, b), \omega > 0)$.

5_1. $f(t,y)$ arb $D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ arb $D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) = 0, u(a) \neq u(b), 4u'(t) > 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) = 0, u(a) \neq u(b), 4u'(t) > 0, t \in (b-\omega, b), \omega > 0)$.

5_2. $f(t,y)$ arb $D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ arb $D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) = 0, u(a) \neq u(b), 4u'(t) < 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) = 0, u(a) \neq u(b), 4u'(t) < 0, t \in (b-\omega, b), \omega > 0)$.

5_3. $f(t,y)$ arb $D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ arb $D^-_{\delta,b}(u))$
for $|u'(a) - u'(b)| = 4u'(a) = 0, u(a) \neq u(b), 4u'(t) \equiv 0, t \in (a, a+\omega), \omega > 0$
$(|u'(a) - u'(b)| = 4u'(b) = 0, u(a) \neq u(b), 4u'(t) \equiv 0, t \in (b-\omega, b), \omega > 0)$.

6. $f(t,y)$ arb $D^+_{\delta,a}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,a}(u)$
$(f(t,y) \geq 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y)$ arb $D^-_{\delta,b}(u))$
for $4u'(a) > |u'(a) - u'(b)| = 0, u(a) \neq u(b)$
$(4u'(b) > |u'(a) - u'(b)| = 0, u(a) \neq u(b))$.

7_1. $f(t,y) \geq 0, (t,y) \in D^+_{\delta,a}(u)$ *and* $f(t,y) \equiv 0, (t,y) \in D^-_{\delta,a}(u)$ $(f(t,y) \equiv 0, (t,y) \in D^+_{\delta,b}(u)$ *and* $f(t,y) \leq 0, (t,y) \in D^-_{\delta,b}(u))$ *for* $|u'(a) - u'(b)| > 0$,

$u'(a) < 0,\ u(a) \neq u(b)\ (|u'(a) - u'(b)| > 0, u'(b) < 0, u(a) \neq u(b))$.

7_{21}. $f(t,y) \geq 0, (t,y) \in D^{+}_{\delta,a}(u)$ and $f(t,y) \equiv 0, (t,y) \in D^{-}_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^{+}_{\delta,b}(u)$ and $f(t,y) \leq 0, (t,y) \in D^{-}_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| > -4u'(a), u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| > -4u'(b), u(a) = u(b))$.

7_{22}. $f(t,y) \geq 0, (t,y) \in D^{+}_{\delta,a}(u)$ and $f(t,y) \geq 0, (t,y) \in D^{-}_{\delta,a}(u)$
$(f(t,y) \leq 0, (t,y) \in D^{+}_{\delta,b}(u)$ and $f(t,y) \leq 0, (t,y) \in D^{-}_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| < -4u'(a), u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| > -4u'(b), u(a) = u(b))$.

7_{231}. $f(t,y) \geq 0, (t,y) \in D^{+}_{\delta,a}(u)$ and $f(t,y) \equiv 0, (t,y) \in D^{-}_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^{+}_{\delta,b}(u)$ and $f(t,y) \leq 0, (t,y) \in D^{-}_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| = -4u'(a)$,
$|u'(a) - u'(b)| > -4u'(t), t \in (a, a+\omega), \omega > 0, u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| = -4u'(b)$,
$|u'(a) - u'(b)| > -4u'(t), t \in (b-\omega, b), \omega > 0, u(a) = u(b))$.

7_{232}. $f(t,y) \geq 0, (t,y) \in D^{+}_{\delta,a}(u)$ and $f(t,y) \geq 0, (t,y) \in D^{-}_{\delta,a}(u)$
$(f(t,y) \leq 0, (t,y) \in D^{+}_{\delta,b}(u)$ and $f(t,y) \leq 0, (t,y) \in D^{-}_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| = -4u'(a)$,
$|u'(a) - u'(b)| < -4u'(t), t \in (a, a+\omega), \omega > 0, u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| = -4u'(b)$,
$|u'(a) - u'(b)| < -4u'(t), t \in (b-\omega, b), \omega > 0, u(a) = u(b))$.

7_{233}. $f(t,y) \geq 0, (t,y) \in D^{+}_{\delta,a}(u)$ and $f(t,y) \geq 0, (t,y) \in D^{-}_{\delta,a}(u)$
$(f(t,y) \leq 0, (t,y) \in D^{+}_{\delta,b}(u)$ and $f(t,y) \leq 0, (t,y) \in D^{-}_{\delta,b}(u))$
for $|u'(a) - u'(b)| > 0, u'(a) < 0, |u'(a) - u'(b)| = -4u'(a)$,
$|u'(a) - u'(b)| \equiv -4u'(t), t \in (a, a+\omega), \omega > 0, u(a) = u(b)$
$(|u'(a) - u'(b)| > 0, u'(b) < 0, |u'(a) - u'(b)| = -4u'(b)$,
$|u'(a) - u'(b)| \equiv -4u'(t), t \in (b-\omega, b), \omega > 0, u(a) = u(b))$.

8. $f(t,y)$ arb $D^{+}_{\delta,a}(u)$ and $f(t,y) \equiv 0, (t,y) \in D^{-}_{\delta,a}(u)$
$(f(t,y) \equiv 0, (t,y) \in D^{+}_{\delta,b}(u)$ and $f(t,y)$ arb $D^{-}_{\delta,b}(u))$ for $|u'(a) - u'(b)| = 0$,
$u'(a) < 0, u(a) \neq u(b)\ (|u'(a) - u'(b)| = 0, u'(b) < 0, u(a) \neq u(b))$.

Then there exists ϵ_0 such that for any $\epsilon \in (0, \epsilon_0]$ the problem $(PBVP_2)$ has solution satisfying the inequality

$$-v_{11} - v_2 - C\epsilon \leq y(t,\epsilon) - u(t) \leq v_{12} + v_2 + C\epsilon$$

on $[a,b]$ where v_{11}, v_{12} and v_2 are the functions from theorem 1 and C is a positive constant.

Proof. The idea of the proof is essentially the same as of theorem 1. Let us define the lower solutions by

$$\alpha(t,\epsilon) = u(t) - v_{11}(t,\epsilon) - v_2(t,\epsilon) - \Gamma(\epsilon)$$

and the upper solutions by

$$\beta(t,\epsilon) = u(t) + v_{12}(t,\epsilon) + v_2(t,\epsilon) + \Gamma(\epsilon).$$

Analogously as in theorem 1 we obtain

$$\epsilon\alpha'' - F(t,\alpha,\alpha') \geq -\epsilon|u''| + \gamma\epsilon - f(t,\alpha)\left(\alpha'^2 - u'^2\right)$$

$$= -\epsilon|u''| + \gamma\epsilon + f(t,\alpha)(v'_{11} + v'_2)(2u' - v'_{11} - v'_2)$$

and

$$F(t,\beta,\beta') - \epsilon\beta'' \geq -\epsilon|u''| + \gamma\epsilon + f(t,\beta)\left(\beta'^2 - u'^2\right)$$

$$= -\epsilon|u''| + \gamma\epsilon + f(t,\beta)(v'_{12} + v'_2)(2u' + v'_{12} + v'_2).$$

Similarly as in previous theorem we conclude that

$$f(t,\alpha)(v'_{11} + v'_2)(2u' - v'_{11} - v'_2) \geq 0$$

$$(f(t,\beta)(v'_{12} + v'_2)(2u' + v'_{12} + v'_2) \geq 0)$$

or

$$|f(t,\alpha)(v'_1 + v'_2)(2u' - v'_{11} - v'_2)| \leq c_2\epsilon$$

$$(|f(t,\beta)(v'_{12} + v'_2)(2u' + v'_{12} + v'_2)| \leq c_2\epsilon)$$

on $[a,b]$ for some positive constant c_2 and sufficiently small ϵ, for instance, when $\epsilon \in (0,\epsilon_0]$. Therefore, for $\gamma \geq c_2 + \max\{|u''(t)|, t \in [a,b]\}$ is $\epsilon\alpha''(t,\epsilon) \geq f(t,\alpha(t,\epsilon))\alpha'^2(t,\epsilon) + g(t,\alpha(t,\epsilon))$ and $\epsilon\beta''(t,\epsilon) \leq f(t,\beta(t,\epsilon))\beta'^2(t,\epsilon) + g(t,\beta(t,\epsilon))$ on $[a,b]$. Hence theorem 2 is proved. □

Remark 2. As a remark we note, that the assumed condition $u(a) \geq u(b)$ is not essential. One can take into consideration the lower and upper solutions for $u(a) \leq u(b)$ and reformulated theorem 2 for this case.

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Faculty of Materials Science and Technology, Department of Mathematics
Paulínska 16, 917 24 Trnava, SLOVAK REPUBLIC
E-mail: vrabel@sun.mtf.stuba.sk