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Archivum Mathematicum, Vol. 36 (2000), No. 1, 17--24

Persistent URL: `http://dml.cz/dmlcz/107714`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 17 – 24

PARTIAL DENSITIES ON THE GROUP OF INTEGERS

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Abstract. Conditions are obtained under which a partial density on the group of integers with the discrete topology can be extended to a density.

1. Introduction

Berg and Rubel (1969) investigated densities on locally commpact abelian (LCA) groups and Neiderreiter and Sookoo [4] obtained conditions under which a partial density on an LCA group can be extended to a density.

In this paper, we obtain additional conditions when the LCA group is the group of integers with the discrete topology. In Section 2, we present notations and definitions and in Section 3 the additional conditions in question.

2. Definitions and Notations

For a compact, Hausdorff space X, u.d. can be defined with respect to a nonnegative, regular, normed Borel measure (c.f. Kuipers and Niederreiter (1974)).

Notation. (i) Let μ_B be a nonnegative, regular, normed, Borel measure on X.

(ii) Let $R(X)$ denote the set of all continuous, real-valued functions on X.

Definition. The sequence (x_n) is u.d. in X with respect to μ_B if

$$\lim_{N\to\infty} \sum_{n=1}^{N} f(x_n) = \int_X f\, d\mu_B \qquad \forall\, f \in R(x)$$

A density (c.f. Berg and Rubel (1969)) on an LCA group G is a system of measures on subgroups of compact index of G satisfying compatibility conditions.

1991 *Mathematics Subject Classification*: 28C10.
Key words and phrases: partial density, extension to density.
Received August 24,1998.

Definition. A closed subgroup H of an LCA group G is said to be *of compact index* if G/H is compact.

Notation. (i) Let $\{H_\alpha | \alpha \in A\}$ be the set of all subgroups of G of compact index, where A is a suitable index set.

(ii) Let $\{G_\alpha | \alpha \in A\}$ be the set of compact quotients of G, where $G_\alpha = G/H_\alpha$ for each $\alpha \in A$.

Definition. A system D of measures given by

$$D = \{\mu_\alpha | \mu_\alpha \quad \text{is a probability measure on} \quad G_\alpha, \ \alpha \in A\}$$

is called a *density* on G if it satisfies the following compatibility condition:

If $\psi : G_\beta \to G_\alpha$ is the natural homomorphism from G_β to a quotient G_α of G_β, then for any Borel set B in G_α, $\mu_\alpha(B) = \mu_\beta(\psi^{-1}(B))$.

We next define a partial density (c.f. Niederreiter (1975)).

Definition. Let G be an LCA group and $\{H_\alpha | \alpha \in A\}$ be the set of all subgroups of compact index of G. For a subset B of A, let

$$P = \{\mu_\alpha | \mu_\alpha \quad \text{is a probability measure on} \quad G_\alpha, \ \alpha \in B\}$$

be a system of measures satisfying the following compatibility condition:

If

$$H_\alpha \supseteq H_{\beta_1} \quad \text{and} \quad H_\alpha \supseteq H_{\beta_2}\,,$$

where $\alpha \in A$, $\beta_1 \in B$ and $\beta_2 \in B$, then μ_{β_1} and μ_{β_2} induce the same measure on G_α. Then P is called a *partial density* on G.

Notation. Let $\mathbb{Z}$be the group of integers with the discrete topology, and $R(\mathbb{Z})$ the set of continuous, real-valued functions on $\mathbb{Z}$

3. Conditions for the Extension of a Partial Density

Lemma 3.1. *If $gcd(m_1, m_2) = 1$, μ_1 is a measure on $\mathbb{Z}/m_1\mathbb{Z}$and μ_2 a measure on $\mathbb{Z}/m_2\mathbb{Z}$, then there exists a measure μ on $\mathbb{Z}/m_1m_2\mathbb{Z}$which induces μ_1 and μ_2.*

Proof. Let μ be the measure on the direct product $(\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$ given by $\mu = \mu_1 \times \mu_2$; that is, μ is the direct product of μ_1 and μ_2.

If $A \in (\mathbb{Z}/m_1\mathbb{Z})$, then

$$\begin{aligned} \mu(A \times \mathbb{Z}/m_2\mathbb{Z}) &= \mu_1(A)\mu_2(\mathbb{Z}/m_2\mathbb{Z}) \\ &= \mu_1(A) \end{aligned}$$

Continuing like this, we see that μ induces μ_1 and μ_2.

Also $\mathbb{Z}/m_1m_2\mathbb{Z} \cong (\mathbb{Z}/m_1\mathbb{Z}) \times (\mathbb{Z}/m_2\mathbb{Z})$ because of the following result:

Let a, b be integers. Then $X \equiv a \mod m_1$ and $X \equiv b \mod m_2$ if and only if $X \equiv r \mod m_1m_2$, where r is an integer uniquely determined $\mod m_1m_2$ by a and b. (This result can be deduced from the Chinese Remainder Theorem.)

Hence μ can be considered as a measure on $\mathbb{Z}/m_1m_2\mathbb{Z}$, and we know that μ is compatible with μ_1 and μ_2. □

Lemma 3.2. *If μ_1 is a measure on $\mathbb{Z}/m_1\mathbb{Z}$ and μ_2 is a measure on $\mathbb{Z}/m_2\mathbb{Z}$ such that $\{\mu_1, \mu_2\}$ is a partial density on $\mathbb{Z}$, then there exists a measure μ on $\mathbb{Z}/I\mathbb{Z}$ which induces μ_1 and μ_2 where I is the L.C.M. of m_1 and m_2.*

Proof. If $d = gcd(m_1, m_2)$, we can reduce the problem to d simpler problems. In each case, we will have two relatively prime numbers, as in Lemma 3.1. The equations involving the measures of the cosets $i + 0$, $i + d$, $i + 2d, \ldots, i + (I - d)$ would not have terms involving the measures of any other cosets, where $i \in \{0, 1, 2, \ldots, d-1\}$. (For a nonnegative integer p less than I, p denotes the coset $\ldots, -I+p, p, I+p, \ldots$ of $I\mathbb{Z}$ in $\mathbb{Z}/I\mathbb{Z}$).

We show this as follows. Let μ_1 take the values $x_0, x_1, \ldots, x_{m_1-1}$ and μ_2 take the values $y_0, y_1, \ldots y_{m_2-1}$. We wish to find a measure μ on $\mathbb{Z}/I\mathbb{Z}$ satisfying the equations

$$\begin{aligned} \mu(0) + \mu(m_1) + \cdots + \mu(I - m_1) &= x_0 \\ \mu(1) + \mu(m_1 + 1) + \cdots + \mu(I - m_1 + 1) &= x_1 \\ &\vdots \\ \mu(m_1 - 1) + \mu(2m_1 - 1) + \cdots + \mu(I - 1) &= x_{m_1} - 1 \end{aligned}$$

and

$$\begin{aligned} \mu(0) + \mu(m_2) + \cdots + \mu(I - m_2) &= y_0 \\ \mu(1) + \mu(m_2 + 1) + \cdots + \mu(I - m_2 + 1) &= y_1 \\ &\vdots \\ \mu(m_2 - 1) + \mu(2m_2 - 1) + \cdots + \mu(I - 1) &= y_{m_2-1} \end{aligned}$$

Because of the compatibility condition on μ_1 and μ_2,

$$\begin{aligned} x_0 + x_d + \cdots + x_{m_1-d} &= y_0 + y_d + \cdots + y_{m_2-d} \\ &= \alpha\,, \quad \text{say}\,. \end{aligned}$$

The set S of eguations involving $\mu(0)$, $\mu(d)$, $\mu(2d), \ldots, \mu(I - d)$ does not involve any other unknowns, so they can be solved separately.

If $\alpha = 0$, we let $\mu(i) = 0$ for $i \in \{0, d, 2d, \ldots, I - d\}$.

If $\alpha > 0$, we multiply $x_0, x_d, \ldots, x_{m_1-d}$, $y_0, y_d, \ldots, y_{m_2-d}$ by $1/\alpha$.

Now consider this problem:

Let ν_1 be a measure on $\mathbb{Z}/\frac{m_1}{d}\mathbb{Z}$ having values $\frac{x_0}{\alpha}, \frac{x_d}{\alpha}, \ldots, \frac{x_{m_1-d}}{\alpha}$ and let ν_2 be a measure on $\mathbb{Z}/\frac{m_2}{d}\mathbb{Z}$ having values $\frac{y_0}{\alpha}, \frac{y_d}{\alpha}, \ldots, \frac{y_{m-2-d}}{\alpha}$.

Then ν_1 and ν_2 are probability measures and $\frac{m_1}{d}$ and $\frac{m_2}{d}$ are relatively prime.

Hence from Lemma 3.1, there exists a measure ν on $\mathbb{Z}/\frac{I}{d}\mathbb{Z}$ which induces ν_1 and ν_2. Hence, if we multiply each equation in S by $1/\alpha$, and then replace $\mu(i)$ by $\nu(\frac{i}{d})$, the new set of equation has at least one solution such that $\nu(\frac{i}{d}) \geq 0$, for $i \in \{0, d, 2d, \ldots, I - d\}$.

Similarly, the equations involving

$$\mu(j), \mu(j+d), \mu(j+2d), \ldots, \mu(j+I-d), \qquad j \in \{1, 2, \ldots, d-1\},$$

have nonnegative solutions.

Hence our original set of equation have nonnegative solutions. Hence there exists a measure μ which induces μ_1 and μ_2.

Lemma 3.3. *Let $\{\mu_{m_i} | i = 1, 2, \ldots, p\}$ be a partial density on $\mathbb{Z}$ Then there exists a measure on $\mathbb{Z}/(L.C.M.$ of $m_1, m_2, \ldots, m_p)\mathbb{Z}$ which induces μ_{m_i} for each i in $\{1, 2, \ldots, p\}$, if the following condition is satisfied for each element a in $\{1, 2, \ldots, p-1\}$:*

Let $R = L.C.M.$ of $m_1, m_2, \ldots, m_a$. Then $gcd(R, m_{a+1})$ is a divisor of at least one of $m_1, m_2, \ldots, m_a$.

Proof. Let the $L.C.M.$ of $m_1, m_2, \ldots, m_a$ be $I^{12\ldots a}$. There is a measure $\mu_{I^{12}}$ on $\mathbb{Z}/I^{12}\mathbb{Z}$ which induces μ_{m_1} and μ_{m_2}, according to Lemma 3.2.

Now, $\dot{T} = gcd(I^{12}, m_3)$ is a divisor of at least one of m_1 and m_2. Hence $u_{I^{12}}$ and μ_{m_3} are compatible with respect to the greatest common divisor, $\dot{T}$. Therefore there exists a measure $u_{I^{123}}$ on $\mathbb{Z}/I^{123}\mathbb{Z}$ compatible with μ_{m_1}, μ_{m_2}, μ_{m_3}.

Now $gcd(I^{123}, m_4)$ is a divisor of at least one of m_1, m_2, m_3. Hence there exists a measure on $\mathbb{Z}/I^{1234}\mathbb{Z}$ compatible with μ_{m_1}, μ_{m_2}, μ_{m_3} and μ_{m_4}.

Continuing like this, we obtain a measure $\mu_{I^{12\ldots p}}$ on $\mathbb{Z}/I^{12\ldots p}\mathbb{Z}$ compatible with μ_{m_i} for each i in $\{1, 2, \ldots, p\}$. □

Theorem 3.4. *Let $P = \{\mu_{m_i} | i \in B\}$ be a partial density on $\mathbb{Z}$ Then P can be extended to a density on $\mathbb{Z}$ if the following condition is satisfied:*

Let $I_{m_1 m_2 \ldots m_a}$ be the $L.C.M.$ of $m_1, m_2, \ldots, m_a$ for arbitrary a in B. Then $gcd(I_{m_1 m_2 \ldots m_a}, m_{a+1})$ is a divisor of at least one of $m_1, m_2, \ldots, m_a$, for each a in B.

Proof. Let N_{m_i} be the set of continuous, real-valued functions on $\mathbb{Z}$ having period m_i for each i in B, and let M be the space of finite linear combinations of elements of the N_{m_i} over $\mathbb{R}, i \in B$. Define L on M as follows:

If $f \in M$, then

$$f = \sum_{i=1}^{n} k_i f_i$$

for some $f_i \in N_{m_i}$ $i = 1, 2, \ldots, n$ and $k_i \in \mathbb{R}$ $i = 1, 2, \ldots, n$.

Let

$$L(f) = \sum_{i=1}^{n} \int_{\mathbb{Z}/m_i\mathbb{Z}} k_i f_i \, d\mu_{m_i}$$

where μ_{m_i} is the measure on $\mathbb{Z}/m_i\mathbb{Z}$ in P; $i \in \{1, 2, \ldots, n\}$.

Now let $f \in M$ such that $f \geq 0$. Then $f = f_1 + f_2 + \cdots + f_a$ for some a in B, where f_i has period m_i, $i \in \{1, 2, \ldots, a\}$. Lemma 3.3 implies that there is a probability measure μ on $\mathbb{Z}/I_{m_1 m_2 \ldots m_a}\mathbb{Z}$ such that μ is compatible with $\mu_1, \mu_2, \ldots, \mu_a$.

Hence

$$\begin{aligned} L(f) &= \sum_{i=1}^{a} \int_{\mathbb{Z}/m_i\mathbb{Z}} f_i \, d\mu_{m_i} \\ &= \sum_{i=1}^{a} \int_{\mathbb{Z}/m_1m_2\ldots m_a\mathbb{Z}} f_i \, d\mu \\ &= \int_{\mathbb{Z}/m_1m_2\ldots m_a\mathbb{Z}} f \, d\mu \geq 0 \end{aligned}$$

since $f \geq 0$.

Hence L is positive; therefore Theorem 3.4 of [4] implies that P can be extended to a density on $\mathbb{Z}$

If the measures in a partial density satisfy a certain condition, then the partial density can be extended to a density on $\mathbb{Z}$ In this case the partial density can be defined for any set of subgroups of $\mathbb{Z}$ The following theorem gives this condition. First we need a definition. □

Definition. Let $m_1, m_2, \ldots, m_n$ be positive integers greater than one and let I be their LCM. If

$$S = \{\nu_\alpha | \alpha = 1, \ldots, n; \nu_\alpha \text{ is a signed measure on } \mathbb{Z}/m_\alpha\mathbb{Z}\}$$

satisfies the usual condition for a partial density, and $\|S\| = 1$, we call S a *signed partial density*.

Theorem 3.5. *Let $I, m_1, m_2, \ldots, m_n$ be as above.*

Suppose that

$$P = \{\mu_\alpha | \mu_\alpha \text{ is a probability measure on } \mathbb{Z}/m_\alpha\mathbb{Z}, \alpha = 1, 2, \ldots, n\}$$

is a partial density on $\mathbb{Z}$such that

$$\mu_\alpha = \frac{\nu_\alpha + (2^{n+1} - 1)I\overline{\mu}_\alpha}{(2^{n+1} - 1)I + 1}$$

where $\{\nu_\alpha | \alpha = 1, 2, \ldots, n\}$ is a signed partial density and $\overline{\mu}_\alpha$ is the Haar measure on $\mathbb{Z}/m_\alpha\mathbb{Z}$, $\alpha = 1, 2, \ldots, n, I$. Then P can be extended to a density on $\mathbb{Z}$ In particular, if

$$\mu_\alpha \geq \frac{(2^{n+1} - 1)I\overline{\mu}_\alpha}{(2^{n+1} - 1)I + 1}$$

for each α in $\{1, 2, \ldots, n\}$, then P can be extended to a density on $\mathbb{Z}$

Proof. We define a linear functional L on the set M of linear combinations over $|R$ of functions from $R(\mathbb{Z}/m_\alpha\mathbb{Z})$, $\alpha = 1, 2, \ldots, n$ as follows:

If $f = f_1 + f_2 + \cdots + f_n$, $f_\alpha \in R(\mathbb{Z}/m_\alpha\mathbb{Z})$, $\alpha \in \{1, 2, \ldots, n\}$

then

$$L(f) = \sum_{\alpha=1}^{n} \int_{\mathbb{Z}/m_\alpha\mathbb{Z}} f_\alpha \, d\nu_\alpha$$

where $R(\mathbb{Z}/m_\alpha\mathbb{Z})$ is the set of real valued, continuous functions on $\mathbb{Z}/m_\alpha\mathbb{Z}$. We define another linear functional Q on M for each α in $\{1, 2, \ldots, n\}$ as follows:

$$Q(f) = \frac{L(f) + (2^{n+1} - 1)I \int_{\mathbb{Z}/I\mathbb{Z}} f \, d\overline{\mu}_I}{(2^{n+1} - 1)I + 1}$$

Then

$$\begin{aligned} Q(f) &= \sum_{\alpha=1}^{n} \frac{\int_{\mathbb{Z}/m_\alpha\mathbb{Z}} f_\alpha \, d\nu_\alpha + (2^{n+1} - 1)I \int_{\mathbb{Z}/m_\alpha\mathbb{Z}} f_\alpha \, d\overline{\mu}_\alpha}{(2^{n+1} - 1)I + 1} \\ &= \sum_{\alpha=1}^{n} \int_{\mathbb{Z}/m_\alpha\mathbb{Z}} f_\alpha \, d\left(\frac{\nu_\alpha + (2^{n+1} - 1)I\overline{\mu}_\alpha}{(2^{n+1} - 1)I + 1}\right) \\ &= \sum_{\alpha=1}^{n} \int_{\mathbb{Z}/m_\alpha\mathbb{Z}} f_\alpha \, d\mu_\alpha \end{aligned}$$

Since $\{\nu_\alpha | \alpha = 1, 2, \ldots, n\}$ is a signed partial density, Lemma 3.3 of [4] implies

$$|L(f)| \leq (2^{n+1} - 1) \text{ for each } f \in M \text{ such that } \|f\| \leq 1.$$

If L is positive, then Q is positive. If L is not positive, let f_0 be the element of M for which L is minimum with $0 \leq f_0 \leq 1$. Then f_0 must take the value 1 on at least one element of $\mathbb{Z}/I\mathbb{Z}$, otherwise there would be a positive multiple f_1 of f_0 such that $L(f_1) < L(f_0)$.

Now $\overline{\mu}_I(P) \geq \frac{1}{I}$ for each P in $\mathbb{Z}/I\mathbb{Z}$ and so

$$\int_{\mathbb{Z}/I\mathbb{Z}} f_0 \, d\overline{\mu}_I \geq \frac{1}{I}.$$

Since

$$\begin{aligned} |L(f_0)| &\geq (2^{n+1} - 1) \\ Q(f_0) &\geq 0 \\ \therefore \quad Q(f) &\geq 0, \ \forall \ f \in M \quad \text{such that } 0 \leq f \leq 1. \end{aligned}$$

Since any positive function in M is a multiple of some $g_0 \in M$ for which $0 \leq g_0 \leq 1$, Q is positive. Since L is bounded, Q is bounded and so continuous. Hence Q can be extended to a positive, continuous linear functional L_1 on $R(\mathbb{Z})$. L_1 induces on G a density which is an extension of P.

Suppose now that

$$\mu_\alpha \geq \frac{(2^{n+1} - 1)I\overline{\mu}_\alpha}{(2^{n+1} - 1)I + 1};$$

then μ_α can be expressed in the form

$$\frac{\nu_\alpha + (2^{n+1}-1)I\overline{\mu}_\alpha}{(2^{n+1}-1)I+1}$$

where $\{\nu_\alpha | \alpha = 1, 2, \ldots, n\}$ is a partial density, and the result follows as before.

In the general case, we cannot use a method similar to the one used in Lemma 3.1. The following case illustrates this.

Let μ_4 and μ_6 be measures on $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$ respectively. Let also $\mu_a(i+b\mathbb{Z})$ be the measure of the coset $i+b\mathbb{Z}$ of $b\mathbb{Z}$ in $\mathbb{Z}/b\mathbb{Z}$ where a and b are positive integers greater than 1, μ_a is some measure defined on $\mathbb{Z}/b\mathbb{Z}$, and $i \in \{0, 1, 2, \ldots, b-1\}$. Suppose that

$$\begin{aligned}
\mu_6(0+6\mathbb{Z}) &= 1/6\\
\mu_6(1+6\mathbb{Z}) &= 0/6\\
\mu_6(2+6\mathbb{Z}) &= 2/6\\
\mu_6(3+6\mathbb{Z}) &= 0/6\\
\mu_6(4+6\mathbb{Z}) &= 2/6\\
\mu_6(5+6\mathbb{Z}) &= 1/6\\
\mu_4(0+4\mathbb{Z}) &= 5/6\\
\mu_4(1+4\mathbb{Z}) &= 1/6\\
\mu_4(2+4\mathbb{Z}) &= 0/6\\
\mu_4(3+4\mathbb{Z}) &= 0/6
\end{aligned}$$

We see that $\{\mu_4, \mu_6\}$ is a partial density on $\mathbb{Z}$

We have measures μ_4 on $\mathbb{Z}/4\mathbb{Z}$ and μ_3 on $\mathbb{Z}/3\mathbb{Z}$, where μ_6 induces μ_3.

$$\begin{aligned}
\mu_3(0+3\mathbb{Z}) &= 1/6\\
\mu_3(1+3\mathbb{Z}) &= 2/6\\
\mu_3(2+3\mathbb{Z}) &= 3/6
\end{aligned}$$

We define μ_{12} on $\mathbb{Z}/12\mathbb{Z}$ by $\mu_{12} = \mu_4 \times \mu_3$. μ_4 and μ_6 induce μ_2 on $\mathbb{Z}/2\mathbb{Z}$ where

$$\begin{aligned}
\mu_2(0+2\mathbb{Z}) &= 5/6\\
\mu_2(1+2\mathbb{Z}) &= 1/6
\end{aligned}$$

But μ_{12} induces $\overline{\mu_6}$ on $\mathbb{Z}/6\mathbb{Z}$, where $\mu_{\overline{6}}$ takes the values

$$\begin{aligned}
1/6 \times 5/6 &= 5/36\\
1/6 \times 1/6 &= 1/36\\
2/6 \times 5/6 &= 10/36\\
2/6 \times 1/6 &= 2/36\\
3/6 \times 5/6 &= 15/36\\
3/6 \times 1/6 &= 3/36
\end{aligned}$$

where

$$\begin{aligned}\overline{\mu}_6(0+6\mathbb{Z}) &= 5/36\\ \overline{\mu}_6(1+6\mathbb{Z}) &= 2/36\\ \overline{\mu}_6(2+6\mathbb{Z}) &= 15/36\\ \overline{\mu}_6(3+6\mathbb{Z}) &= 1/36\\ \overline{\mu}_6(4+6\mathbb{Z}) &= 10/36\\ \overline{\mu}_6(5+6\mathbb{Z}) &= 3/36\end{aligned}$$

since $\overline{\mu}_6$ nust induce μ_3 and μ_2. We see that $\overline{\mu}_6$ is different from μ_6. Hence, while μ_{12} is compatible with μ_2 and μ_3 it is not compatible with μ_6.

Therefore the method used in Lemma 3.1 is not suitable in the general case.

Remark. The following result can be shown using some of the previous theorems.

Let G be an LCA group such that the periodic characters form a countable subgroup of $\widehat{G}$. Let S and I be collections of subgroups of compact index of G such that:

(i) Finite intersections of members of $S\ U\ I$ are in $S\ U\ I$.

(ii) For each H in S, μ_H is the Haar measure on G/H.

If, for each K in I, there exists a probability measure (other than the Haar measure) on G/K, such that $\{\mu_H|H \in S\}\ U\{\mu_k|K \in I\}$ is a partial density on G, then there exists a sequence (g_n) in G such that (g_nH) is u.d. in G/H for each H in S, but (g_nK) is not u.d. in G/K for any K in I.

This can be shown as follows:

The partial density can be extended to a density on G. By Theorem 5 of [3], there exists a sequence (g_n) in G such that (g_nH) is u.d. in G/H with respect to μ_H for each H in S and (g_nK) is u.d. in G/K with respect to μ_K for each K in I. Hence (g_nH) is u.d. in G/H (with respect to Haar measure on G/H) for each H in S and (g_nK) is not u.d. in G/K (with respect to the Haar measure on G/K) for each K in I.

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