Aleka Kalapodi; Angeliki Kontolatou; Jiří Močkoř
Some properties of Lorenzen ideal systems

*Archivum Mathematicum*, Vol. 36 (2000), No. 4, 287--295

Persistent URL: `http://dml.cz/dmlcz/107743`

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 287 – 295

# SOME PROPERTIES OF LORENZEN IDEAL SYSTEMS

A. KALAPODI, A. KONTOLATOU AND J. MOČKOŘ

Abstract. Let $G$ be a partially ordered abelian group (*po*-group). The construction of the Lorenzen ideal $r_a$-system in $G$ is investigated and the functorial properties of this construction with respect to the semigroup $(R(G), \oplus, \leq)$ of all $r$-ideal systems defined on $G$ are derived, where for $r, s \in R(G)$ and a lower bounded subset $X \subseteq G$, $X_{r\oplus s} = X_r \cap X_s$. It is proved that Lorenzen construction is the natural transformation between two functors from the category of *po*-groups with special morphisms into the category of abelian ordered semigroups.

## 1. Introduction

The investigation of arithmetical properties of partly ordered groups (*po*-groups) has its origin in the study of arithmetics of integral domains. An important example of such arithmetical property is the notion of the Kronecker function ring in the theory of divisibility of integral domains. This notion was introduced by W. Krull in order to study the arithmetics of integral domains. The principal advantage of the extension process which leads from an integrally closed domain $A$ to its Kronecker function ring $K(A)$ is the fact that $K(A)$ is the Bezout domain, i.e. any finitely generated ideal is principal. This problem of embedding of an integral domain into a greatest common divisor integral domain (GCD-domain) is in the centre of many arithmetical problems and in the course of time it has become more and more clear that all these arithmetical notions in integral domains have their purely multiplicative analogues in (commutative) semigroups with cancellation law, or equivalently, in *po*-groups. It seems that the principal tool for the investigation of these properties in *po*-groups is the notion of an $r$-ideal which has its origin in a paper of Lorenzen [7]. We recall that by an $r$-system of ideals in a directed *po*-group $G$ we mean a map $X \mapsto X_r$ ($X_r$ is called an $r$-ideal) from the set $B(G)$ of all lower bounded subsets $X$ of $G$ into the power set of $G$ which satisfies the following conditions:

(1) $X \subseteq X_r$
(2) $X \subseteq Y_r \Rightarrow X_r \subseteq Y_r$

---

1999 *Mathematics Subject Classification*: 06F05, 06F20.
*Key words and phrases*: $r$-ideal, $r_a$-system, system of finite character.
Received November 19, 1999.

(3) $\{a\}_r = a \cdot G^+ = (a)$ for all $a \in G$
(4) $a \cdot X_r = (a \cdot X)_r$ for all $a \in G$.

The theory of $r$-ideals of *po*-groups seems to be a tool which enables us to establish relationships between arithmetical properties of integral domains and the theory of *po*-groups.

For any $r$-system $r$ on $G$ by $I_r(G)$ we denote the set of all $r$-ideals on $G$. This set is a commutative monoid with respect the operation $\times_r$, where $X_r \times_r Y_r = (X \cdot Y)_r = (X_r \cdot Y_r)_r$ for all $X_r, Y_r \in I_r(G)$.

The principal problem of embedding of an integral domain into a GCD-domain can be translated into the problem of embedding of a *po*-group into a lattice-ordered group ($l$-group). And it is well known that from $r$-ideal systems point of view this problem is connected with *Lorenzen ideal systems*. Recall that for any $r$-closed *po*-group $G$ with an $r$-system (i.e. $X_r : X_r \subseteq G^+$ for any finite $X \subseteq G$), we can associate to the $r$-system $r$ another $r$-system denoted by $r_a$ such that

$$(\forall X \in B(G) \text{ finite }), X_{r_a} = \{g \in G : g \cdot K_r \subseteq X_r \times_r K_r, \text{ for some finite } K \subseteq G\},$$

$$(\forall X \in B(G)), X_{r_a} = \bigcup_{\substack{K \subseteq X \\ K \text{ finite}}} K_{r_a}.$$

The principal property of this $r$-system $r_a$ is that the monoid of finitely generated $r_a$-ideals (under $r_a$-multiplication $\times_{r_a}$) satisfies the cancellation law and hence possesses the quotient group $\Lambda_r(G)$ which is called the *Lorenzen $r$-group of $G$*. This group is a lattice ordered group if we set $\Lambda_r(G)^+ = \{A_{r_a}/B_{r_a} : A_{r_a} \subseteq B_{r_a}\}$ and, moreover, $\Lambda_r(G)$ contains $G$ as an ordered subgroup. Hence the Lorenzen $r$-system admits the principal role in investigation of the arithmetical properties.

In this note we investigate some properties of this Lorenzen $r$-system $r_a$. We are interested mostly in functorial properties of the Lorenzen map $a : r \mapsto r_a$. For a *po*-group $G$ by $R(G)$ we denote the set of all $r$-systems defined on $G$ and, moreover, we set:

(1) $R_\alpha(G) = \{r \in R(G) : (\forall X \in B(G))X_r = (a) \text{ for some } a \in G\}$.
(2) $R_\beta(G) = \{r \in R(G) : (\forall X \in B(G))(\exists Y \in B(G))X_r \times_r Y_r = (1_G)\}$,
(3) $R_\gamma(G) = \{r \in R(G) : (\forall X, Y, Z \in B(G))X_r \times_r Y_r = X_r \times_r Z_r \Rightarrow Y_r = Z_r\}$,
(4) $R_\delta(G) = \{r \in R(G) : (\forall X \in B(G), X \text{ finite })X_r : X_r \subseteq G^+\}$,

where for every $A, B \subseteq G$, we set $A : B = \{x \in G : x \cdot B \subseteq A\}$ and $G^+$ is the positive cone of $G$. Ideal systems from these sets have special importance for investigation of arithmetical properties of *po*-groups. By $R_{fin}(G)$ we denote the set of $r$-systems on $G$ which are of finite character, i.e. for any $X \in B(G)$ and $x \in X_r$, there exists a finite set $K \subseteq X$ such that $x \in K_r$, that is, $X_r = \bigcup_{\substack{K \subseteq X \\ K \text{ finite}}} K_r$.On the set $R(G)$ an ordering can be defined such that $r \leq s \Leftrightarrow X_s \subseteq X_r$, for each $X \in B(G)$. If $r \leq s$, we say that $s$ is finer than $r$ or that $r$ is coarser than $s$. Among all $r$-systems in $R(G)$, there exists one, called the $v$-system, which is the coarsest one and it is defined by $X_v = \bigcap_{X \subseteq (x)} (x)$, for each $X \in B(G)$.

In this paper we show that $a = a(G) : R_\delta(G) \to R_\gamma(G)$ is a homomorphism of ordered semigroups with respect to the semigroup operation $\oplus$ defined on $R(G)$

such that $X_{r\oplus s} = X_r \cap X_s$ for $r, s \in R_\delta(G)$ and $X \in B(G)$. Moreover, we prove that this homomorphism is a $(v, v)$-homomorphism, i.e. for any $W \subseteq R_\delta(G)$ we have $a(W_v) \subseteq (a(W))_v$, where $W_v$ is the $v$-ideal generated by $W$ in the semigroup $(R_\delta(G), \oplus)$. Another functorial character of Lorenzen map is connected with the map $-_H : R_{fin}(G) \to R_{fin}(G/H)$, where $H$ is a directed convex subgroup of a *po*-group $G$. This map was introduced in [9] and it seems to be useful for investigation of arithmetical properties of quotient integral domains $D_S$, where $S$ is a multiplicative system in $D$. In this paper we prove that Lorenzen map commutes with this map $-_H$, i.e. we prove that $-_H\cdot a(G) = a(G/H)\cdot -_H$. From this it follows that Lorenzen map $a$ can be considered as a special natural transformation between two functors. Namely, if $\mathcal{G}$ is the category of directed *po*-groups with morphisms $G \to G/H$ such that $H$ is an $o$-ideal of $G$ and if $\mathcal{S}$ is the category of abelian ordered semigroups with corresponding homomorphisms as morphisms then we can find two functors $\mathbf{R}_\delta, \mathbf{R}_\gamma : \mathcal{G} \to \mathcal{S}$ such that $\mathbf{R}_\delta(G) = (R_\delta(G), \oplus, \leq)$ and $\mathbf{R}_\gamma(G) = (R_\gamma(G), \oplus, \leq)$. Then Lorenzen map defines the natural transformation $\mathbf{a} : \mathbf{R}_\delta \to \mathbf{R}_\gamma$.

## 2. Lorenzen $r_a$-systems

We start this section with the following simple propositions which extend results of Halter-Koch [4].

**Proposition 2.1.** *Let $G$ be a directed po-group. For $r, s \in R(G)$ and $X \in B(G)$ we set $X_{r\oplus s} = X_r \cap X_s$. Then $r \oplus s \in R(G)$ and $(R(G), \oplus, \leq)$ becomes a partially ordered semigroup, which is a semilattice.*

The proof can be done analogously as for $r$-systems in semigroups and it will be omitted (see [4]).

**Proposition 2.2.** *Let $G, H$ be two directed groups and let $u : G \to H$ be an order isomorphism. Then the semigroups $(R(G), \oplus, \leq)$ and $(R(H), \oplus, \leq)$ are order-isomorphic.*

**Proof.** Let be $X \in B(H)$. We set

$$\bar{u} : R(G) \to R(H), r \mapsto \bar{u}(r),$$
$$X_{\bar{u}(r)} = u\left(\left(u^{-1}(X)\right)_r\right) .$$

The map $\bar{u}$ is well defined (see [4], p.48). Let be $r, s \in R(G)$. Since

$$X_{\bar{u}(r\oplus s)} = u\left(\left(u^{-1}(X)\right)_r \cap \left(u^{-1}(X)\right)_s\right) = X_{\bar{u}(r)\oplus\bar{u}(s)} ,$$

there holds $\bar{u}\,(r \oplus s) = \bar{u}(r) \oplus \bar{u}(s)$. Now let $w \in R(H)$. We put

$$r : B(G) \to 2^G,\ Z_r = u^{-1}\left((u(Z))_w\right) .$$

Obviously, $r$ belongs to $R(G)$ and

$$X_{\bar{u}(r)} = u\left(u^{-1}\left(\left(u\left(u^{-1}(X)\right)\right)_w\right)\right) = u\left(u^{-1}\left(X_w\right)\right) = X_w ,$$

which means that $\bar{u}(r) = w$. The rest of statements can be proved simply. Thus, the map $\bar{u}$ is an order semigroup isomorphism. □

**Proposition 2.3.** *Let $G$ be a directed po-group.*

(1) *The structure $(R_\alpha(G), \oplus, \leq)$ is a partially ordered semigroup, if and only if, $G$ is an $l$-group.*
(2) *The structure $(R_\beta(G), \oplus, \leq)$ is a partially ordered semigroup.*
(3) *The structure $(R_\gamma(G), \oplus, \leq)$ is a partially ordered semigroup.*
(4) *The structure $(R_\delta(G), \oplus, \leq)$ is a partially ordered semigroup.*

**Proof.** (1) Let $G$ be an $l$-group. The set $R_\alpha(G)$ is non-empty, since, for $X \subseteq G$ finite, we can set $X_r = (\wedge X)$. Let $r, s \in R_\alpha(G)$. Then, for any finite $X \in B(G)$, there holds

$$X_{r\oplus s} = X_r \cap X_s = \{a\}_r \cap \{b\}_s = \{a \vee b\}_{r\oplus s}\,,$$

that is, $r \oplus s \in R_\alpha(G)$ and $R_\alpha(G)$ is a subsemigroup of $R(G)$. Conversely, if $(R_\alpha(G), \oplus)$ is a semigroup, then for any $a, b \in G$ and any $r \in R_\alpha(G)$, we have $\{a, b\}_r = \{c\}_r$ for some $c \in G$ and it is clear that, in this case, $c = a \wedge b$. Thus, $G$ is an $l$-group.

(2) It is well known that $R_\beta(G) = \{v\}$ (see [5]). But in this case we have $v \oplus v = v$.

(3) Let $r, s \in R_\gamma(G)$ and let $X, Y \in B(G)$ be finite subsets, such that $Y_{r\oplus s} \subseteq X_{r\oplus s} \times_{r\oplus s} Y_{r\oplus s}$. Then, $Y \subseteq (X \cdot Y)_r$ and $Y \subseteq (X \cdot Y)_s$ and we have $Y_r \subseteq (X \cdot Y)_r$ and $Y_s \subseteq (X \cdot Y)_s$. Thus, according to [5], $1 \in X_r \cap X_s = X_{r\oplus s}$, which means that $r \oplus s \in R_\gamma(G)$.

(4) It follows directly from [5],p.25. □

As we mentioned in the introduction for any $r$-system $r \in R_\delta(G)$ we can construct another $r$-system $r_a$ which belongs to $R_\gamma(G)$, in the following way: for every finite $X \in B(G)$, the element $x$ belongs to $X_{r_a}$, if and only if, there exists a finite $r$-ideal $K_r$, such that $x \cdot K_r \subseteq X_r \times_r K_r$. It is clear that $r_a \leq r$ and that for every $r, s \in R_\delta(G)$, with $r \leq s$, it follows $r_a \leq s_a$. Moreover, $r = r_a$, if and only if, $r \in R_\gamma(G)$. There also holds $(r_a)_a = r_a$ (see [4]).

We are interested in functorial properties of this map $a = a(G) : R_\delta(G) \to R_\gamma(G)$. The following proposition describes such property.

**Proposition 2.4.** *Let $G$ be a directed po-group. Then the map $a = a(G) : (R_\delta(G), \oplus, \leq) \to (R_\gamma(G), \oplus, \leq)$ ,$a(r) = r_a$, is a homomorphism of ordered semigroups.*

**Proof.** Let be $r, s \in R_\delta(G)$. Since $r \oplus s \in R_\delta(G)$, the ideal system $(r \oplus s)_a$ is well defined. We have to prove that $(r \oplus s)_a = r_a \oplus s_a$. There holds $r_a \leq r$, so $r_a \oplus s_a \leq r \oplus s$ and it follows that

$$r_a \oplus s_a \leq (r \oplus s)_a\,.$$

Conversely, let us consider a finite $X \in B(G)$ and $x \in X_{r_a\oplus s_a}$. Then, there exist finite $Y, Z \in B(G)$, such that $x \cdot Y_r \subseteq (X \cdot Y)_r$ and $x \cdot Z_s \subseteq (X \cdot Z)_s$. Put $K = Y \cdot Z$. Then,

$$x \cdot K_r = x \cdot (Y \cdot Z)_r = (x \cdot Y)_r \times_r Z_r \subseteq (X \cdot Y)_r \times_r Z_r = (X \cdot K)_r$$

and similarly $x \cdot K_s \subseteq (X \cdot K)_s$. Thus,

$$x \cdot K_{r\oplus s} = x \cdot K_r \cap x \cdot K_s \subseteq (X \cdot K)_r \cap (X \cdot K)_s = (X \cdot K)_{r\oplus s}\,,$$

which means that $x \in X_{(r\oplus s)_a}$. Hence,

$$(r \oplus s)_a \leq r_a \oplus s_a\,. \qquad \square$$

According to [4], we can also define an ideal system on a semigroup. A very natural one is the $v$-system. Recall that for a semigroup $(S,*)$ and $A \subseteq S$ such that $A \subseteq b * S$ for some $b \in S$ ($A$ is then called lower bounded), the $v$-ideal generated by $A$ in $S$ is the subset

$$A_v = \bigcap_{\substack{b\in S\\ A\subseteq b*S}} b * S\,.$$

In a classical way we can then introduce the notion of the $(v,v)$-morphism between semigroups with $v$-systems defined on them. Hence if $(S,s)$ and $(T,t)$ are semigroups with $r$-systems defined then a homomorphism $f : S \to T$ is called a $(s,t)$-morphism if for any lower bounded $X \subseteq S$ we have $f(X_s) \subseteq (f(X))_t$.

**Proposition 2.5.** *The semigroup homomorphism $a : (R_\delta(G),\oplus) \to (R_\gamma(G),\oplus)$ is a $(v,v)$-morphism, that is, for any lower bounded $W \subseteq R_\delta(G)$, we have $a(W_v) \subseteq (a(W))_v$.*

**Proof.** Let $W \subseteq R_\delta(G)$ be lower bounded and let $s \in W_v$. We prove that

$$s_a \in (a(W))_v = \bigcap_{\substack{r\in R_\gamma(G)\\ a(W)\subseteq r\oplus R_\gamma(G)}} r \oplus R_\gamma(G)\,.$$

Let $r \in R_\gamma(G)$ be such that $a(W) \subseteq r \oplus R_\gamma(G)$. Then, for any $w \in W$ there exists $\bar{w} \in R_\gamma(G)$, such that $w_a = r \oplus \bar{w}$. Since $w_a \leq w$, it follows that

$$w = w \oplus w_a = r \oplus (\bar{w} \oplus w)\,,$$

where $\bar{w} \oplus w \in R_\delta(G)$. Hence, $W \subseteq r \oplus R_\delta(G)$. Since

$$s \in W_v = \bigcap_{\substack{u\in R_\delta(G)\\ W\subseteq u\oplus R_\delta(G)}} u \oplus R_\delta(G)\,,$$

it holds $s = r \oplus t$ for some $t \in R_\delta(G)$. Then,

$$s_a = (r \oplus t)_a = r_a \oplus t_a = r \oplus t_a\,,$$

thus, $s_a \in r \oplus R_\gamma(G)$. Therefore, $s_a \in (a(W))_v$. $\square$

Given two systems $r_1 \in R(G_1)$ and $r_2 \in R(G_2)$ we can define an ideal system on the cartesian product $G = G_1 \times G_2$, symbolized by $r_1 \otimes r_2$, as follows: For every $X \in B(G)$, put $X_{r_1\otimes r_2} = (p_1(X))_{r_1} \times (p_2(X))_{r_2}$, where $p_i$, $i = 1,2$, are the usual projection maps.

In [6] it is proved that $r_i \in R_j(G_i)$, for $i = 1,2$ and $j = \gamma,\delta$ respectively, implies $r_1 \otimes r_2 \in R_j(G_1 \times G_2)$, $j = \gamma,\delta$ respectively. Thus, for $j = \gamma,\delta$ respectively, we obtain a map

$$- \otimes - : R_j(G_1) \times R_j(G_2) \to R_j(G_1 \times G_2)\,.$$

**Proposition 2.6.** *Let $G_1, G_2$ be two directed po-groups and $G = G_1 \times G_2$. Then the following diagram commutes*

$$\begin{array}{ccc} R_\delta(G_1) \times R_\delta(G_2) & \xrightarrow{-\otimes-} & R_\delta(G) \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle a(G)} \\ R_\gamma(G_1) \times R_\gamma(G_2) & \xrightarrow{-\otimes-} & R_\gamma(G) \end{array}$$

*where $a(r,s) = (r_a, s_a)$.*

**Proof.** Let us consider a finite $X \in B(G)$ and $x = (x_1, x_2) \in G$. Let be $x \in X_{r_a \otimes s_a} = (p_1(X))_{r_a} \times (p_2(X))_{s_a}$. Then, there exist finite $Y_1 \in B(G_1), Y_2 \in B(G_2)$, such that

$$x_1 \cdot (Y_1)_r \subseteq (p_1(X))_r \times_r (Y_1)_r \text{ and } x_2 \cdot (Y_2)_s \subseteq (p_2(X))_s \times_s (Y_2)_s \,.$$

Put $Y = Y_1 \times Y_2$. Then,

$$x \cdot Y_{r\otimes s} = x_1 \cdot (Y_1)_r \times x_2 \cdot (Y_2)_s \subseteq (X \cdot Y)_{r\otimes s} \,.$$

and it follows that $(r \otimes s)_a \le r_a \otimes s_a$.

Conversely, let $x \in X_{(r\otimes s)_a}$. Then, there exists a finite $Y \subseteq G_1 \times G_2$, such that $x \cdot Y_{r\otimes s} \subseteq (X \cdot Y)_{r\otimes s}$. Then, $x_1 \cdot (p_1(Y))_r \subseteq (p_1(X) \cdot p_1(Y))_r$ and $x_2 \cdot (p_2(Y))_s \subseteq (p_2(X) \cdot p_2(Y))_s$ and it follows that $x \in (p_1(X))_{r_a} \times (p_2(X))_{s_a} = X_{r_a \otimes s_a}$. Hence, $(r \otimes s)_a = r_a \otimes s_a$. □

Recall that an $r$-system $r$ on $G$ is called of *finite character* if, for any $X \in B(G)$ we have $X_r = \bigcup\limits_{\substack{K \subseteq X \\ K \text{ finite}}} K_r$. By $R_{fin}(G)$ we denote the subset of $R(G)$ consisting of all $r$-systems of finite character.

**Lemma 2.7.** *The structure $(R_{fin}(G), \oplus, \le)$ is a partially ordered semigroup.*

**Proof.** Let be $r, s \in R_{fin}(G)$ and $x \in X_{r\oplus s}$, $X \in B(G)$. Then, there exist finite subsets $K, L \subseteq X$ such that $x \in K_r \cap L_s$. Put $M = K \cup L$. Hence, $M$ is a finite subset of $X$ and $x \in M_r \cap M_s = M_{r\oplus s}$. Therefore, $r \oplus s$ is of finite character. □

Let $H$ be a directed convex subgroup of a directed *po*-group $G$ with an $r$-system $r$ (i.e. $H$ is an *o-ideal of* $G$) and let $f : G \longrightarrow G/H$, $f(a) = aH$ for $a \in G$, be the canonical homomorphism. Then for any lower bounded subset $\mathcal{A} \subseteq G/H$ we may find a lower bounded subset $A \subseteq G$ such that $\{aH : a \in A\} = \mathcal{A}$. In fact, see [9], if $A$ is any subset of $G$ representing $\mathcal{A}$, then for a lower bound $aH = \alpha$ of $\mathcal{A}$ , for any $\beta \in \mathcal{A}$ there exist $h_\beta \in H$ and $b_\beta \in A$ such that $a \le b_\beta \cdot h_\beta, b_\beta H = \beta$. Hence, $\{b_\beta \cdot h_\beta : \beta \in \mathcal{A}\}$ is a lower bounded set representing $\mathcal{A}$. Then we set

$$\mathcal{A}_{r_H} = A_r / H \,.$$

**Lemma 2.8 rm (See [9]).** *$r_H$ is an $r$-system of finite character on $G/H$.*

**Proof.** It is clear that we have to prove only that this definition is correct. Hence, let $B$ be another lower bounded set of representants of $\mathcal{A}$ and let $\sigma : A \longrightarrow B$ be such that $aH = \sigma(a)H$ for all $a \in A$. Let $a \in A_r$. Since $r$ is of finite character, there exists a finite subset $K \subseteq A$ such that $a \in K_r$. Then for all $k \in K$ there exists $h_k \in H$ such that $k = \sigma(k) \cdot h_k$. Since $H$ is an $o$-ideal, there exists $h \in H$ such that $h \leq h_k$ for all $k \in K$. Hence, $k \geq \sigma(k)h$ and it follows that $k \in \{\sigma(k)h : k \in K\}_r = h \cdot \{\sigma(k) : k \in K\}_r$. Thus, $K_r \subseteq h \cdot \{\sigma(k) : k \in K\}_r$ and $aH \in B_r/H$. The rest of the proof may be done simply. □

In this way, for any directed *po*-group $G$ and any $o$-ideal $H$ of $G$, we obtain a map

$$-_H : R_{fin}(G) \to R_{fin}(G/H)\,, \quad r \mapsto r_H\,.$$

**Proposition 2.9.** *Let $G$ be a directed po-group and let $H$ be an o-ideal of $G$. Then the map*

$$-_H : (R_{fin}(G), \oplus, \leq) \to (R_{fin}(G/H), \oplus, \leq)$$

*is a homomorphism of ordered semigroups.*

**Proof.** Let be $r, s \in R_{fin}(G)$. It is enough to prove that $(r \oplus s)_H = r_H \oplus s_H$, since it is clear that from $r \leq s$ it follows that $r_H \leq s_H$. Let us consider $\mathcal{A} \in B(G/H)$ and $A \in B(G)$, such that $\mathcal{A} = A/H$. Then,

$$\mathcal{A}_{(r\oplus s)_H} = A_{r\oplus s}/H = (A_r \cap A_s)\,/H\,,$$
$$\mathcal{A}_{r_H \oplus s_H} = \mathcal{A}_{r_H} \cap \mathcal{A}_{s_H} = A_r/H \cap A_s/H\,.$$

Obviously, $(A_r \cap A_s)\,/H \subseteq A_r/H \cap A_s/H$, which means that $r_H \oplus s_H \leq (r \oplus s)_H$.

Conversely, let be $gH \in A_r/H \cap A_s/H$. Then, there exist $x_1 \in A_r, x_2 \in A_s$, such that $gH = x_1H = x_2H$ and there are $h_1, h_2 \in H$, such that $x_1 = g \cdot h_1, x_2 = g \cdot h_2$. Since $H$ is directed, there exists $h \in H$, such that $h \geq h_1, h_2$ and therefore $g \cdot h \geq x_1, x_2$. Thus,

$$g \cdot h \in \{x_1\}_r \cap \{x_2\}_s \subseteq A_r \cap A_s\,,$$

hence, $gH = (g \cdot h)H \in (A_r \cap A_s)\,/H$, that is $(r \oplus s)_H \leq r_H \oplus s_H$. □

We set $R_{\delta,fin}(G) = R_\delta(G) \cap R_{fin}(G)$ and $R_{\gamma,fin}(G) = R_\gamma(G) \cap R_{fin}(G)$. It follows from the previous results that these sets are semigroups with respect to the operation $\oplus$.

**Theorem 2.10.** *Let $G$ be a directed po-group and let $H$ be an o-ideal of $G$. Then the following diagram commutes*

$$\begin{tikzcd} R_{\delta,fin}(G) \arrow[r, "a(G)"] \arrow[d, "-_H"'] & R_{\gamma,fin}(G) \arrow[d, "-_H"] \\ R_{\delta,fin}(G/H) \arrow[r, "a(G/H)"] & R_{\gamma,fin}(G/H) \end{tikzcd}$$

**Proof.** We show at first that this diagram is correct, i.e. that for any $r \in R_{\delta,fin}(G)$, we have $r_H \in R_{\delta,fin}(G/H)$ and analogously for $r \in R_{\gamma,fin}(G)$. Let $\mathcal{A} \subseteq G/H$ be a finite set and let $\alpha \in G/H$ be such that $\alpha \cdot \mathcal{A}_{r_H} \subseteq \mathcal{A}_{r_H}$. Let $A$ be a finite set such that $\mathcal{A}_{r_H} = A_r/H$, $A_r = \{a_1, \dots, a_n\}_r$, $\alpha = aH$. Since $a \cdot a_i H \in A_r/H$, for any $i$ there exist $g_i \in A_r$ and $h_i \in H$ such that $a \cdot a_i = g_i \cdot h_i$. Since $H$ is directed, there exists $h \in H$ such that $h \leq h_i$ for all $i$. Then we have

$$\begin{aligned}\{a \cdot a_1, \dots, a \cdot a_n\}_r &= \{g_1 \cdot h_1, \dots, g_n \cdot h_n\}_r \subseteq \{g_1 \cdot h, \dots, g_n \cdot h\}_r \\ &= h \cdot \{g_1, \dots, g_n\}_r \subseteq h \cdot \{a_1, \dots, a_n\}_r ,\end{aligned}$$

and it follows that $ah^{-1} \geq 1$. Hence, $\alpha \geq 1_{G/H}$ and $r_H \in R_{\delta,fin}(G/H)$.

Now, let $r \in R_{\gamma,fin}(G)$. According to [5]; Lemma, par.2, Chapt.2, we have only to prove that for any finitely generated $r_H$-ideals $\mathcal{A}_{r_H}, \mathcal{B}_{r_H}$ in $G/H$ from $\mathcal{A}_{r_H} \subseteq \mathcal{B}_{r_H} \times \mathcal{A}_{r_H}$ it follows that $1_{G/H} \in \mathcal{B}_{r_H}$. Let $\mathcal{A}_{r_H} = A_r/H$, $\mathcal{B}_{r_H} = B_r/H$. If $\mathcal{A}_{r_H} \subseteq (\mathcal{A} \cdot \mathcal{B})_{r_H}$, then for any $a \in A$ we have $aH \in (A \cdot B)_r/H$ and for any $a_i \in A = \{a_1, \dots, a_n\}$ there exist $c_i \in (A \cdot B)_r$ and $h_i \in H$ such that $a_i = c_i \cdot h_i$. Since $H$ is directed there exists $h \in H$ such that $h \leq h_i$ for all $i$. Then

$$\begin{aligned}A_r &= \{c_1 \cdot h_1, \dots, c_n \cdot h_n\}_r \subseteq \{c_1, \dots, c_n\}_r \cdot h \subseteq (A \cdot B)_r \cdot h \\ &= A_r \times_r (B \cdot h)_r .\end{aligned}$$

Since $r \in R_\gamma(G)$ , we have $1 \in (B \cdot h)_r$ and it follows that $1_{G/H} \in B_r/H$. Finally, we show that the diagramm commutes. Let $r \in R_{\delta,fin}(G)$. Then we have to prove that $(r_a)_H = (r_H)_a$. Let $\mathcal{A} \subseteq G/H$ be a finite subset and let $A \subseteq G$ be a finite subset such that $\mathcal{A} = A/H$. Then we have

$$\begin{aligned}\mathcal{A}_{(r_H)_a} &= \{gH \in G/H : \exists \mathcal{K} \subseteq G/H \text{ finite }, gH \cdot \mathcal{K}_{r_H} \subseteq \mathcal{A}_{r_H} \times_{r_H} \mathcal{K}_{r_H}\} , \\ \mathcal{A}_{(r_a)_H} &= \{gH \in G/H : \exists K \subseteq G \text{ finite }, g \cdot K_r \subseteq A_r \times_r K_r\} .\end{aligned}$$

Let $gH \in \mathcal{A}_{(r_a)_H}$ and let $g \cdot K_r \subseteq A_r \times_r K_r$ for some $K \subseteq G$ finite. Then we have $gH \cdot (K_r/H) \subseteq (A \cdot K)_r/H = A_r/H \times_{r_H} K_r/H$ and it follows that $gH \in \mathcal{A}_{(r_H)_a}$.

Conversely, let $gH \in \mathcal{A}_{(r_a)_H}$ and let $gH \cdot \mathcal{K}_{r_H} \subseteq \mathcal{A}_{r_H} \times_{r_H} \mathcal{K}_{r_H}$ for some $\mathcal{K} \subseteq G/H$ finite. Let $\mathcal{K}_{r_H} = K_r/H$ for some $K \subseteq G$ finite. Then we have $(g \cdot K)_r/H = gH \cdot K_r/H \subseteq (A \cdot K)_r/H$. Hence, for any $b \in K$ there exists $h_b \in H$ and $y_b \in (A \cdot K)_r$ such that $h_b \cdot g \cdot b = y_b$. Since $H$ is directed, there exists $h \in H$ such that $h \geq h_b$ for all $b \in K$ and it follows that $g \cdot h \cdot b \geq y_b$ for all $b \in K$. Therefore, we have $g \cdot h \cdot b \in \{y_b\}_r \subseteq (A \cdot K)_r$ and for $a = g \cdot h$ we have $a \cdot K \subseteq (A \cdot K)_r$. Therefore, $a \cdot K_r \subseteq A_r \times_r K_r$, $aH = gH$ and $gH \in \mathcal{A}_{(r_a)_H}$. □

Let $\mathcal{G}$ be the category of directed *po*-groups with morphisms $G \to G/H$ such that $H$ is an *o*-ideal of $G$ and let $\mathcal{S}$ be the category of abelian ordered semigroups with corresponding homomorphisms as morphisms. It is clear that the definition of $\mathcal{G}$ is correct since the composition of two morphisms in $\mathcal{G}$ is again a morphism in $\mathcal{G}$. The following corollary then follows from Theorem 2.10

**Corollary.** *There exist two functors* $\mathbf{R}_\delta, \mathbf{R}_\gamma : \mathcal{G} \to \mathcal{S}$ *such that* $\mathbf{R}_\delta(G) = (R_\delta(G), \oplus, \leq)$ *and* $\mathbf{R}_\gamma(G) = (R_\gamma(G), \oplus, \leq)$. *Moreover, Lorenzen map* $a : r \mapsto r_a$ *defines the natural transformation* $\mathbf{a} : \mathbf{R}_\delta \to \mathbf{R}_\gamma$.

**Proof.** Let $f : G \to G/H$ be a morphism in $\mathcal{G}$. Then we put $\mathbf{R}_\alpha(u)(r) = r_H$ for $r \in R_\alpha(G)$ and $\alpha = \delta, \gamma$. $\square$

## References


[1] Arnold, I., *Ideale in kommutativen Halbgruppen*, Rec. Math. Soc. Math. Moscow **36** (1929), 401–407.

[2] Borewicz, I. R., Shafarevicz, Z. I., *Number Theory*, Academic Press, New York, 1966.

[3] Clifford, A. H., *Arithmetic and ideal theory of abstract multiplication*, Ann. of Math. **39** (1938), 594–610.

[4] Halter-Koch, F., *Ideal systems*, Marcel Dekker, Inc, New York - Basel - Hong Hong, 1998.

[5] Jaffard, P., *Les systémes d'idéaux*, Dunod, Paris, 1960.

[6] Kalapodi, A. and Kontolatou, A., *Algebraic and categorical properties of r-ideal systems*, International Journal of Mathematics and Mathematical Sciences, to appear.

[7] Lorenzen, P., *Abstrakte Begründung der multiplikativen Idealtheorie*, Math. Z. **45** (1939), 533–553.

[8] Močkoř, J., *Groups of Divisibility*, D. Reidl Publ. Co., Dordrecht, 1983.

[9] Močkoř, J., Kontolatou, A., *Groups with quasi divisor theory*, Comm. Math. Univ. St. Pauli, Tokyo **42** (1993), 23–36.

[10] Skula, L., *Divisorentheorie einer Halbgruppe*, Math. Z. **114** (1970), 113–120.



Aleka Kalapodi
Department of Mathematics, University of Patras
26500 Patras, GREECE
*E-mail*: Kalapodi@math.upatras.gr

Angeliki Kontolatou
Department of Mathematics, University of Patras
26500 Patras, GREECE
*E-mail*: angelika@math.upatras.gr

Jiří Močkoř
Department of Mathematics, University of Ostrava
CZ-702 00 Ostrava, Bráfova 7, CZECH REPUBLIC
*E-mail*: Mockor@osu.cz