Miroslav Doupovec; Ivan Kolář
Natural transformations of separated jets

Archivum Mathematicum, Vol. 36 (2000), No. 4, 297--303

Persistent URL: http://dml.cz/dmlcz/107744
NATURAL TRANSFORMATIONS OF SEPARATED JETS

MIROSLAV DOUPOVEC, IVAN KOĽAŔ

ABSTRACT. Given a map of a product of two manifolds into a third one, one can define its jets of separated orders \( r \) and \( s \). We study the functor \( J^{r:s} \) of separated \((r; s)\)-jets. We determine all natural transformations of \( J^{r:s} \) into itself and we characterize the canonical exchange \( J^{r:s} \to J^{s:r} \) from the naturality point of view.

Let \( M, N, Q \) be manifolds. Given a map \( f : M \times N \to Q \), M. Kawaguchi introduced the concept of jet of separated orders \( r \) and \( s \), [1], see also [5]. Write \( J^{r:s}(M, N, Q) \) for the bundle of all such separated \((r; s)\)-jets. In [2] the second author reformulated the Kawaguchi’s idea in a way that clarifies there is a canonical exchange diffeomorphism \( \kappa_{M,N,Q} : J^{r:s}(M, N, Q) \to J^{s:r}(N, M, Q) \). Let \( \mathcal{M}_f \) be the category of all manifolds and all smooth maps and \( \mathcal{M}_f^m \) be the category of \( m \)-dimensional manifolds and their local diffeomorphisms. In Section 2 we interpret \( J^{r:s} \) as a functor on the product category \( \mathcal{M}_f^m \times \mathcal{M}_f^m \times \mathcal{M}_f \) similarly as the construction of classical \( r \)-jets is viewed as a functor on the category \( \mathcal{M}_f^m \times \mathcal{M}_f \) in [3]. Then \( \kappa \) is a natural equivalence \( J^{r:s} \to J^{s:r} \).

Our main problem is that of uniqueness of \( \kappa \) from the viewpoint of the theory of natural operations, [3]. In Proposition 4 we deduce that for \( r \geq 2 \), \( s \geq 2 \), \( \kappa \) is the only natural equivalence \( J^{r:s} \to J^{s:r} \) over the canonical exchange functor \( \mathcal{M}_f^m \times \mathcal{M}_f^m \times \mathcal{M}_f \to \mathcal{M}_f^m \times \mathcal{M}_f^m \times \mathcal{M}_f \). For \( r = 1 \) or \( s = 1 \), the vector bundle structure of the classical first order jet bundles comes into play in a simple way. In order to prove Proposition 4, we determine all natural transformations \( J^{r:s} \to J^{r:s} \) in Section 3. Here we use essentially a result from [4] that describes all natural transformations of the classical \( r \)-jet functor into itself.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [3].

1999 Mathematics Subject Classification: 58A2.

Key words and phrases: separated jet, canonical exchange, natural transformation.

The authors were supported by a grant of the GA ČR No. 201/99/0296.

Received December 13, 1999.
1. Separated \((r; s)\)-jets

Consider three manifolds \(M, N, Q\), two integers \(r, s\) and a point \((x, y)\) \(\in M \times N\). For every map \(f : M \times N \to Q\), denote by \(f_u : N \to Q\) or \(f_v : M \to Q\) the partial map \(v \mapsto f(u, v)\) or \(u \mapsto f(u, v)\) respectively, \(u \in M, v \in N\). If we construct the \(r\)-jet \(j^r_x f_v\) for every \(v \in N\), we obtain a map \(N \to J^r_x(M, Q)\). Let \(g : M \times N \to Q\) be another map.

**Definition 1.** We say that \(f\) and \(g\) determine the same jet of separated orders \(r\) and \(s\) at \((x, y) \in M \times N\), if

\[
j^r_y(j^r_x f_v) = j^s_y(j^r_x g_v) \in J^r_y(N, J^r_x(M, Q)).
\]

The equivalence class will be denoted by \(j^r_{x,y} f\). In short, \(j^r_{x,y} f\) will be called the separated \((r; s)\)-jet of \(f\) at \((x, y)\).

Consider some local coordinates \(x^i\) on \(M\), \(y^p\) on \(N\) and \(z^a\) on \(Q\), \(i = 1, \ldots, m = \dim M, p = 1, \ldots, n = \dim N, a = 1, \ldots, q = \dim Q\). Write \(\alpha\) or \(\beta\) for a multiindex corresponding to \(x^i\) or \(y^p\), respectively. Let \(f^a(x^i, y^p)\) be the coordinate expression of \(f\). Since the coordinate form of \(j^r_x f_v\) is determined by \(D^a\), \(0 \leq |\alpha| \leq r\), we have

**Proposition 1.** \(j^r_{x,y} f = j^r_{x,y} g\) is characterized by

\[
D_{\alpha\beta} f^a(x, y) = D_{\alpha\beta} g^a(x, y), \quad 0 \leq |\alpha| \leq r, \quad 0 \leq |\beta| \leq s.
\]

Write \(J^{r;s}(M, N, Q)\) for the space of all separated \((r; s)\)-jets of \(M \times N\) into \(Q\). This is a fibered manifold over \(M \times N \times Q\) with the induced coordinates

\[
z^a_{\alpha\beta}, \quad |\alpha| \leq r, \quad |\beta| \leq s.
\]

Analogously to the classical case, \(J^{r;s}_{x,y}(M, N, Q)_z \subset J^{r;s}(M, N, Q)\) means the subset of all separated \((r; s)\)-jets with source \((x, y)\) and target \(z\), \(x \in M, y \in N, z \in Q\).

For every \(\tau \leq r\) and \(\overline{s} \leq s\), we have a canonical projection

\[
z^{r,s}_{x,y} : J^{r;s}(M, N, Q) \to J^{r,s}_{x,y}(M, N, Q).
\]

Write \(\varepsilon : M \times N \to N \times M\) for the exchange map \(\varepsilon(x, y) = (y, x)\). Using (2) we find that \(j^{s,r}_{y,x} (f \circ \varepsilon)\) is determined by \(j^{r,s}_{x,y} f\). This defines a canonical exchange diffeomorphism

\[
z_{M,N,Q} : J^{r;s}(M, N, Q) \to J^{s;r}(N, M, Q).
\]

**Example 1.** For \(M = N = \mathbb{R}, x = y = 0, r = s = 1\) we have \(J^1_{\mathbb{R}}(\mathbb{R}, J^1_{\mathbb{R}}(\mathbb{R}, Q)) = T(TQ)\). In this case, the restriction of \(z^{r,s}_{\mathbb{R}}\) coincides with the well known canonical involution on \(TTQ\).
2. The functor $J^{r:s}$

Consider another manifold $\overline{Q}$.

Lemma 1. Let $g : Q \to \overline{Q}$ be a map and $X = j^{r:s}_{x,y}f \in J^{r:s}(M, N, Q)$. Then $j^{r:s}_{x,y}(g \circ f) \in J^{r:s}(M, N, \overline{Q})$ depends on $j^{r+s}_{f(x,y)}g$ and $X$ only.

Proof. In coordinates, the derivatives in question of $g \circ f$ depend on the derivatives of $g$ up to order $r+s$ and on $X$ only. □

Thus, for every $W \in J^{r+s}_{x,y}(Q, \overline{Q})_w$ and every $X \in J^{r:s}_{x,y}(M, N, Q)_z$, we have defined a composition

$$W \circ X \in J^{r:s}_{x,y}(M, N, \overline{Q})_w.$$ (5)

In the same way, we deduce

Lemma 2. Let $g : \overline{M} \to M$ and $h : \overline{N} \to N$ be two maps, $g(\overline{x}) = x$, $h(\overline{y}) = y$, $x, y \in \overline{M}$, $\overline{y} \in \overline{N}$ and $X = j^{r:s}_{x,y}f \in J^{r:s}(M, N, Q)$. Then $j^{r:s}_{x,y}(f \circ (g \times h)) \in J^{r:s}_{x,y}(\overline{M}, \overline{N}, Q)$ depends on $j^{r:s}_{g}g$, $j^{r+s}_{h}h$ and $X$ only. □

Thus, for $Y \in J^{r:s}_{x,y}(\overline{M}, M)_x$, $Z \in J^{s}_{y,\overline{y}}(\overline{N}, N)_y$ and $X \in J^{r:s}_{x,y}(M, N, Q)_z$ we have defined the composition

$$X \circ (Y, Z) \in J^{r:s}_{x,y}(\overline{M}, \overline{N}, Q)_z.$$ (6)

If we combine both (5) and (6), we obtain

$$W \circ X \circ (Y, Z) \in J^{r:s}_{x,y}(\overline{M}, \overline{N}, \overline{Q})_w.$$ (7)

The associativity properties of (7) follow directly from the associativity of the composition of maps.

Consider two local diffeomorphisms $g : M \to \overline{M}$, $h : N \to \overline{N}$ and a map $f : Q \to \overline{Q}$. Then we define

$$J^{r:s}(g, h, f) : J^{r:s}(M, N, Q) \to J^{r:s}(\overline{M}, \overline{N}, \overline{Q})$$ (8)

by setting, for every $X \in J^{r:s}_{x,y}(M, N, Q)_z$, $g(x) = \overline{x}$, $h(y) = \overline{y}$,

$$J^{r:s}(g, h, f)(X) = (j^{r+s}_{\overline{x},\overline{y}}f) \circ X \circ ((j^{r:s}_{\overline{x},\overline{y}}g)^{-1}, j^{r+s}_{\overline{y}}h^{-1})),$$ (9)

where $g^{-1}$ and $h^{-1}$ are constructed locally.

Clearly, using the terminology of [3], we obtain

Proposition 2. $J^{r:s}$ is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f$.

Remark 1. It is interesting to discuss the order of $J^{r:s}$. In general, a bundle functor $F$ on the product $C_1 \times \cdots \times C_k$ of $k$ categories over manifolds will be called of order $(r_1, \ldots, r_k)$, if for every two $k$-tuples of $C_i$-morphisms $f_i, g_i : A_i \to B_i$, $i = 1, \ldots, k$, the conditions $j^{r_i}_{x_i}f_i = j^{r_i}_{x_i}g_i$, $x_i \in A_i$, imply

$$F(f_1, \ldots, f_k)|_{x_1, \ldots, x_k}(A_1, \ldots, A_k) = F(g_1, \ldots, g_k)|_{x_1, \ldots, x_k}(A_1, \ldots, A_k).$$ (10)

In our case, the order of $J^{r:s}$ is $(r, s, r+s)$. □
3. Natural transformations $J^{r;8} \to J^{r;8}$

In the case of the classical $r$-jet functor $J^r$, which is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, the following list of all natural transformations $J^r \to J^r$ is deduced in [4]. For a map $f : M \to N$, let $f_x^0 : x \in M$, denote the constant map $f_x^0(u) = x$. The so-called contraction $\sigma_{M,N} : J^r(M,N) \to J^r(M,N)$ is defined by

$$\sigma_{M,N}(j^r_x f) = j^r_x(f^0_x).$$

For $r \geq 2$, all natural transformations $J^r \to J^r$ are

$$\text{id}_{J^r(M,N)} \quad \text{and} \quad \sigma_{M,N}. \quad (11)$$

For $r = 1$, $J^1(M,N) = T^*M \otimes TN$ is a vector bundle and all natural transformations $J^1 \to J^1$ are the homotheties

$$k \text{id}_{J^1(M,N)} , \quad k \in \mathbb{R}. \quad (12)$$

Having a map $f : M \times N \to Q$, we define $f_{x,y}^i : M \times N \to Q$, $x \in M$, $y \in N$, $i = 0, 1, 2$, by

$$f^0_{x,y}(u,v) = f(x,y), \quad f^1_{x,y}(u,v) = f(x,v), \quad f^2_{x,y}(u,v) = f(u,y).$$

Then we introduce the following three natural transformations

$$\varrho^0_{M,N,Q}(j^{r;8}_x f) = j^{r;8}_x f^0_x \quad (\text{the total contraction}),$$

$$\varrho^1_{M,N,Q}(j^{r;8}_x f) = j^{r;8}_x f^1_x \quad (\text{the first contraction}),$$

$$\varrho^2_{M,N,Q}(j^{r;8}_x f) = j^{r;8}_x f^2_x \quad (\text{the second contraction}).$$

For $s = 1$ (the case $r = 1$ is quite similar), we can construct further natural transformations as follows. We recall

$$J^{r;1}(M,N,Q) = \bigcup_{x \in M} J^1(N, J^r_x(M,Q)).$$

Take any natural transformation $\tau_{M,Q} : J^r(M,Q) \to J^r(M,Q)$, see (11) or (12). Consider the restriction

$$(\tau_{M,Q})_x : J^r_x(M,Q) \to J^r_x(M,Q), \quad x \in M,$$

and construct the induced jet map

$$J^1(\text{id}_N, (\tau_{M,Q})_x) : J^1(N, J^r_x(M,Q)) \to J^1(N, J^r_x(M,Q)).$$

Taking into account all $x \in M$, we obtain a map

$$J^1_N \tau_{M,Q} : J^{r;1}(M,N,Q) \to J^{r;1}(M,N,Q).$$

Applying further a homothety with coefficient $k \in \mathbb{R}$ on each vector bundle $J^1(N, J^r_x(M,Q))$, we obtain a natural transformation

$$k J^1_N \tau_{M,Q} : J^{r;1}(M,N,Q) \to J^{r;1}(M,N,Q).$$

For $r \geq 2$, the only two possibilities are (11). For $r = 1$, we have $\tau_{M,Q} = k \text{id}_{J^1(M,Q)}$, $k \in \mathbb{R}$.

From the technical point of view, our main result is the following assertion.
**Proposition 3.** All natural transformations \( J^{r;s} \to J^{r;s} \) are 

(i) for \( r \geq 2, s \geq 2 \)

\[
\varrho^0, \varrho^1, \varrho^2, \text{id},
\]

(ii) for \( s = 1, r \geq 2 \) (and analogously for \( r = 1, s \geq 2 \))

\[
kJ^1 \sigma, kJ^1 \text{id}, \quad k \in \mathbb{R},
\]

(iii) for \( r = 1, s = 1 \)

\[
kJ^1 \text{id}, \quad kJ^1 \text{id}, \quad k \in \mathbb{R}.
\]

**Proof.** First of all we discuss the subcategory \( \mathcal{M}f_q \subset \mathcal{M}f \). Applying Lemma 14.11 from [3] to each factor of \( \mathcal{M}f_m \times \mathcal{M}f_n \times \mathcal{M}f_q \), we deduce that every natural transformation of \( J^{r;s} \) into itself is over the identities on bases. Write \( G = G^r \times G^s \times G^{r+s} \) and \( L^{r;s} = J^{r;s}_0(\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^q)_0 \). According to the general theory, [3], we are looking for \( G \)-equivariant maps of \( L^{r;s} \) into itself. By (3), the canonical coordinates on \( L^{r;s} \) are

\[
z_{\alpha\beta}^a, \quad |\alpha| \leq r, \quad |\beta| \leq s, \quad (\alpha, \beta) \neq (0,0).
\]

The action of \( G^1 \times G^1 \times G^1_q \subset G \) on (19) is tensorial. Any smooth map \( f : L^{r;s} \rightarrow L^{r;s} \) is of the form

\[
\tilde{z}_{\alpha\beta} = f^a_{\alpha\beta} (z_{\gamma\delta}^b),
\]

where \( \gamma \) or \( \delta \) is a multiindex corresponding to \( x^i \) or \( y^p \), respectively, and \( b = 1, \ldots, q \). By the homogeneous function theorem, [3], p. 213, the homotheties in \( G^1_m \) yield \( f^a_{\alpha\beta} \) is linear in \( z_{\gamma\delta}^b \). Then the homotheties in \( G^1_m \) and \( G^1_n \) imply that \( f^a_{\alpha\beta} \) depends on \( z_{\gamma\delta}^b \) with \( |\alpha| = |\gamma|, \quad |\beta| = |\delta| \) only. Using the generalized invariant tensor theorem, [3], p. 230, we obtain

\[
\tilde{z}_{\alpha\beta}^a = k_{|\alpha|, |\beta|} z_{\alpha\beta}^a, \quad k_{|\alpha|, |\beta|} \in \mathbb{R}.
\]

Now we proceed by induction with respect to \( r+s \). For \( r+s = 1 \), (21) reads

\[
\tilde{z}_i^a = k_{i,0} z_i^a, \quad \tilde{z}_p^a = k_{0,1} z_p^a.
\]

Consider the kernel \( K \) of the jet projection \( G^{r+s}_q \rightarrow G^{r+s-1}_q \) together with the units of \( G^r_m \) and \( G^s_n \). Hence the canonical coordinates on \( K \) are

\[
A_{b_1 \ldots b_{r+s}}^a.
\]
symmetric in all subscripts. Since the action of $G$ on $L_{m,n,q}^{r,s}$ is given by the jet composition, we have, provided we write explicitly $\alpha = (i_1, \ldots, i_r)$, $\beta = (p_1, \ldots, p_s)$,

$$\Xi_{i_1 \ldots i_r p_1 \ldots p_s}^a = z_{i_1 \ldots i_r p_1 \ldots p_s}^a + A_{b_1 \ldots b_r b_{r+1} \ldots b_{r+s}}^a z_{i_1 \ldots i_r p_1 \ldots p_s}^{b_1} \ldots z_{i_1 \ldots i_r p_1 \ldots p_s}^{b_{r+s}} + \ldots,$$

while the other coordinates on $L_{m,n,q}^{r,s}$ are unchanged. The equivariancy of (21) with $|\alpha| = r$, $|\beta| = s$ with respect to (23) reads

$$k_{r,s} z_{i_1 \ldots i_r p_1 \ldots p_s}^a + k_{1,0} k_{0,1} A_{b_1 \ldots b_r b_{r+1} \ldots b_{r+s}}^a z_{i_1 \ldots i_r p_1 \ldots p_s}^{b_1} \ldots z_{i_1 \ldots i_r p_1 \ldots p_s}^{b_{r+s}} = k_{r,s} (z_{i_1 \ldots i_r p_1 \ldots p_s}^a + A_{b_1 \ldots b_r b_{r+1} \ldots b_{r+s}}^a z_{i_1 \ldots i_r p_1 \ldots p_s}^{b_1} \ldots z_{i_1 \ldots i_r p_1 \ldots p_s}^{b_{r+s}}).$$

This implies

$$k_{r,s} = k_{1,0}^r k_{0,1}^s.$$

The action of $G$ on the subspace $(z_\alpha^0)$ or $(z_\beta^0)$, i.e. $|\beta| = 0$ or $|\alpha| = 0$, respectively, corresponds to the classical jet case. Thus, for $r \geq 2$, $s \geq 2$, (11) yields the following four possibilities

$$|k_{1,0}| = 0, 1, \quad |k_{0,1}| = 0, 1.$$

Then (25) leads to the coordinate form of the four possibilities of (i). For $r \geq 2$ and $s = 1$, (11) and (12) yield $k_{1,0} = 0, 1$, $k_{0,1} = k \in \mathbb{R}$. Then (25) implies (ii). For $r = 1$ and $s = 1$, (12) yields $k_{1,0} = k$, $k_{0,1} = k$. Then (25) implies (iii).

To extend our result from the subcategory $Mf_q$ to the whole category $Mf$, it suffices to consider naturality with respect to the canonical injections $\mathbb{R}^q \to \mathbb{R}^{q+1}$ for all $q$.

\section{The uniqueness of $\varphi$}

In general, consider three categories $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and a functor $\varphi : \mathcal{C} \to \mathcal{D}$. A natural transformation over $\varphi$ of two functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ means a natural transformation $F \to G \circ \varphi$.

In our case, $J^{r:s}$ is a functor on $Mf_m \times Mf_n \times Mf$. Denote by $E$ the exchange functor $E : Mf_m \times Mf_n \times Mf \to Mf_n \times Mf_m \times Mf$, $E(M,N,Q) = (N,M,Q)$, $E(g,h,f) = (h,g,f)$. Then the canonical exchange $\varphi : J^{r:s} \to J^{s:r}$, see (4), is a natural equivalence over $E$.

Let $\tau : J^{r:s} \to J^{s:r}$ be a natural transformation over $E$. Then $\varphi^{-1} \circ \tau$ is a natural transformation $J^{r:s} \to J^{r:s}$ over the identity of $Mf_m \times Mf_n \times Mf$. These are listed in Proposition 3. Thus, we have deduced

\textbf{Proposition 4.} All natural transformations $J^{r:s} \to J^{s:r}$ over $E$ are

(i) $\varphi$, $\varphi \circ g^0$, $\varphi \circ g^1$, $\varphi \circ g^2$ for $r \geq 2$, $s \geq 2$,

(ii) $\varphi \circ kJ^1 \sigma$, $\varphi \circ kJ^1 \text{id}$, $k \in \mathbb{R}$ for $r \geq 2$, $s = 1$,

(iii) $\varphi \circ kJ^1 \text{id}$, $k, \bar{k} \in \mathbb{R}$ for $r = 1$, $s = 1$.

In particular, for $r \geq 2$, $s \geq 2$, $\varphi$ is the only natural equivalence $J^{r:s} \to J^{s:r}$ over $E$. \qed
References


MIROSLAV DOUPOVEC

DEPARTMENT OF MATHEMATICS
FSI VUT BRNO
TECHNICKÁ 2, 616 69 BRNO
CZECH REPUBLIC
E-mail: doupovec@mat.fme.vutbr.cz

IVAN KOLÁŘ

DEPARTMENT OF ALGEBRA AND GEOMETRY
MASARYK UNIVERSITY
JANÁČKOVÉ NÁM. 2A, 662 95 BRNO
CZECH REPUBLIC
E-mail: kolar@math.muni.cz