Partha Guha
Ito equation as a geodesic flow on $\text{Diff}^s(S^1) \circ \text{C}^\infty(S^1)$

*Archivum Mathematicum*, Vol. 36 (2000), No. 4, 305--312

Persistent URL: [http://dml.cz/dmlcz/107745](http://dml.cz/dmlcz/107745)
ITO EQUATION AS A GEODESIC FLOW ON $\text{Diff}^s(S^1) \circledast C^\infty(S^1)$

PARTHA GUHA

Abstract. The Ito equation is shown to be a geodesic flow of $L^2$ metric on the semidirect product space $\text{Diff}^s(S^1) \circledast C^\infty(S^1)$, where $\text{Diff}^s(S^1)$ is the group of orientation preserving Sobolev $H^s$ diffeomorphisms of the circle. We also study a geodesic flow of a $H^1$ metric.

1. Introduction

It is known that the periodic Korteweg-de Vries (KdV) equation can be interpreted as geodesic flow of the right invariant metric on the Bott-Virasoro group, which at the identity is given by the $L^2$-inner product [15,18].

Recently Misiolek [14] and others [7,12,17] showed that an analogous correspondence can be established for the Camassa-Holm equation [4]. It gives rise to a geodesic flow of a certain right invariant Sobolev metric $H^1$ on the Bott-Virasoro group.

Thus we see the KdV and the Camassa-Holm equations arise in a unified geometric construction, both are integrable systems which describe geodesic flows on the Bott-Virasoro group. Earlier it was known that both the KdV and the Camassa-Holm are obtained from different regularisations of the Euler equation for a one dimensional compressible fluid. The Euler equation, of course, describes geodesic motion on the group of orientation preserving diffeomorphisms of the circle $\text{Diff}(S^1)$ with respect to $L^2$ metric [6].

Following Ebin-Marsden [6] we enlarge $\text{Diff}(S^1)$ to a Hilbert manifold $\text{Diff}^s(S^1)$, the diffeomorphism of Sobolev class $H^s$. This is a topological space. If $s > n/2$, it makes sense to talk about an $H^s$ map from one manifold to another. Using local charts, one can check whether the derivation of order $\leq s$ are square integrable.

The Lie algebra of $\text{Diff}^s(S^1) \circledast C^\infty(S^1)$ has a three dimensional extension (explained in the next section)

$$\text{Vect}^s(S^1) \circledast C^\infty(S^1) \oplus \mathbb{R}^3.$$
Then a typical element of this algebra would be

\[(f \frac{d}{dx}, u(x), \alpha) \text{ where } f \frac{d}{dx} \in \text{Vect}(S^1), \ u(x) \in C^\infty(S^1) \ \alpha \in \mathbb{R}^3.\]

The \(\widehat{\text{Diff}}^s(S^1) \circ C^\infty(S^1)\) is the non-trivial extension of \(\text{Diff}^s(S^1) \circ C^\infty(S^1)\).

In this paper we study a geodesic flow on the \(\widehat{\text{Diff}}^s(S^1) \circ C^\infty(S^1)\), which at the identity is given by the \(L^2\) inner product, is a completely integrable coupled nonlinear third order partial differential equation introduced by M. Ito [9]. Hence the Ito equation arises as a geodesic flow, in general of course, these flows are not integrable.

Then we study a geodesic flow of the right invariant inner metric on the \(\widehat{\text{Diff}}^s(S^1) \circ C^\infty(S^1)\), which at the identity is given by the \(H^1\) inner product. Thus we obtain a new coupled nonlinear integrable system. The relation between the Ito and this new system is the same as the relation between the KdV and the Camassa-Holm equations.

Now we state our main result:

**Theorem 1.** Let \(t \mapsto \hat{c}\) be a curve in the \(\widehat{\text{Diff}}^s(S^1) \circ C^\infty(S^1)\). Let \(\hat{c} = (e, e, 0)\) be the initial point, directing to the vector \(\hat{c}(0) = (u(x) \frac{d}{dx}, v(x), \gamma_0)\), where \(\gamma_0 \in \mathbb{R}^3\). Then \(\hat{c}(t)\) is a geodesic of the

(A) \(L^2\) metric if and only if \((u(x,t) \frac{d}{dx}, v(x,t), \gamma)\) satisfies the Ito equation

\[
\begin{align*}
 u_t + u_{xxx} + 6uu_x + 2vv_x &= 0, \\
 v_t + 2(uv)_x &= 0, \\
 \gamma_t &= 0.
\end{align*}
\]

(B) \(H^1\) metric if and only if \((u(x,t) \frac{d}{dx}, v(x,t), \gamma)\) satisfies

\[
\begin{align*}
 u_t - u_{xxx} + uu_{xxx} + 2u_xu_{xx} - uu_{xx} + v_xv_{xx} - 5uu_x - vv_x &= 0, \\
v_t - v_{xx} + uv_{xxx} - uv_x + uu_{xx} - uu_xv &= 0.
\end{align*}
\]

The Ito system [9] admits a bi-Hamiltonian structure

\[
\mathbf{D}_2 \delta H_n = \mathbf{D}_1 \delta H_{n+1},
\]

where

\[
\mathbf{D}_2 = \left( \begin{array}{cc} D^3 + 4uD + 2u_x & 2vD \\ 2v_x + 2vD & 0 \end{array} \right),
\]

\[
\mathbf{D}_1 = \left( \begin{array}{cc} D & 0 \\ 0 & D \end{array} \right)
\]

with the Hamiltonian functionals

\[(1) \quad H_1[u,v] = \frac{1}{2} \int (u^2 + v^2) dx \]

\[(2) \quad H_2[u,v] = \frac{1}{2} \int (u^3 - \frac{1}{2}u_x^2 + uv^2) dx.\]
The recursion operator arising from a Hamiltonian pair
\[ R = D_2D_1^{-1} = \begin{pmatrix} D^2 + 4u + 2u_xD^{-1} & 2v \\ 2v_xD^{-1} + 2v & 0 \end{pmatrix} \]
is a hereditary operator yields infinitely many conserved quantities [1].

**Acknowledgement:** The author is grateful to Mathematische Physik- Technische Universität Clausthal, where the part of the work has been done. This work is partially supported by S. Chandrasekhar Memorial ICSC-World Laboratory fellowship.

2. **ITO EQUATION AND \( L^2 \) METRIC**

Let \( Diff^s(S^1) \) be the group of orientation preserving Sobolev \( H^s \) diffeomorphisms of the circle. It is known that the group \( Diff^s(S^1) \) as well as its Lie algebra of vector fields on \( S^1 \), \( T_{td}Diff^s(S^1) = Vect^s(S^1) \), have non-trivial one-dimensional central extensions, the Bott-Virasoro group \( \hat{Diff}^s(S^1) \) and the Virasoro algebra \( Vir \) respectively [10,11,18].

The Lie algebra \( Vect^s(S^1) \) is the algebra of smooth vector fields on \( S^1 \). This satisfies the commutation relations
\[ [f \frac{d}{dx}, g \frac{d}{dx}] := (f(x)g'(x) - f'(x)g(x)) \frac{d}{dx}. \]

One parameter family of \( Vect^s(S^1) \) acts on the space of smooth functions \( C^\infty(S^1) \) by
\[ L^{(\mu)} f(x) \frac{d}{dx}a(x) = f(x)a'(x) - \mu f'(x)a(x), \]
where
\[ L^{(\mu)} f(x) \frac{d}{dx} = f(x) \frac{d}{dx} - \mu f'(x) \]
is the derivative with respect to the vector field \( f(x) \frac{d}{dx} \).

The Lie algebra of \( Diff^s(S^1) \odot C^\infty(S^1) \) is the semidirect product Lie algebra
\[ G = Vect^s(S^1) \odot C^\infty(S^1). \]

An element of \( G \) is a pair \( (f(x) \frac{d}{dx}, a(x)) \), where \( f(x) \frac{d}{dx} \in Vect^s(S^1) \) and \( a(x) \in C^\infty(S^1) \).

It is known that this algebra has a three dimensional central extension given by the non-trivial cocycles
\[ \omega_1((f \frac{d}{dx}, a), (g \frac{d}{dx}, b)) = \int_{S^1} f'(x)g''(x)dx \]
\[ \omega_2((f \frac{d}{dx}, a), (g \frac{d}{dx}, b)) = \int_{S^1} f''(x)b(x) - g''a(x)dx \]
\[ \omega_3((f \frac{d}{dx}, a), (g \frac{d}{dx}, b)) = 2 \int_{S^1} a(x)b'(x)dx. \]
The first cocycle $\omega_1$ is the well known Gelfand-Fuchs cocycle. The Virasoro algebra is the unique non-trivial central extension of $\text{Vect}(S^1)$ via this $\omega_1$ cocycle. Hence we define the Virasoro algebra
\[
\text{Vir} = \text{Vect}^*(S^1) \oplus \mathbb{R}.
\]
The space $C^\infty(S^1) \oplus \mathbb{R}$ is identified with a part of the dual space to the Virasoro algebra. It is called the regular part, and the pairing between this space and the Virasoro algebra is given by:
\[
\langle (u(x), a), (f(x)\frac{d}{dx}, \alpha) \rangle = \int_{S^1} u(x)f(x)dx + a\alpha.
\]

Similarly we consider an extension of $\mathcal{G}$. This extended algebra is given by
\[
\hat{\mathcal{G}} = \text{Vect}^*(S^1) \odot C^\infty(S^1) \oplus \mathbb{R}^3.
\]

The Lie algebra $\hat{\mathcal{G}}$ has been considered in various places [2,8,13]. It was shown in [16] that the cocycles define the universal central extension the Lie algebra $\text{Vect}^*(S^1) \odot C^\infty(S^1)$. This means $H^2(\text{Vect}^*(S^1) \odot C^\infty(S^1)) = \mathbb{R}^3$.

**Definition 1.** The commutation relation in $\hat{\mathcal{G}}$ is given by
\[
[(f\frac{d}{dx}, a, \alpha), (g\frac{d}{dx}, b, \beta)] := ((fg' - f'g)\frac{d}{dx}, fb' - ga', \omega)
\]
where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$, $\omega = (\omega_1, \omega_2, \omega_3)$ are the two cocycles.

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on $S^1$. Of particular interest are the orbits in $\hat{\mathcal{G}}^*_{\text{reg}}$. In the case of current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

**Definition 2.** The regular part of the dual space $\hat{\mathcal{G}}^*$ to the Lie algebra $\hat{\mathcal{G}}$ as follows: Consider
\[
\hat{\mathcal{G}}^*_{\text{reg}} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3.
\]
and fix the pairing between this space and $\hat{\mathcal{G}}$, $\langle \cdot, \cdot \rangle : \hat{\mathcal{G}}^*_{\text{reg}} \otimes \hat{\mathcal{G}} \to \mathbb{R}$:
\[
\langle \hat{u}, \hat{f} \rangle = \int_{S^1} f(x)u(x)dx + \int_{S^1} a(x)v(x)dx + \alpha \cdot \gamma,
\]
where $\hat{u} = (u(x), v, \gamma)$, $\hat{f} = (f\frac{d}{dx}, a, \alpha)$.

Extend (8) to a right invariant metric on the semi-direct product space $\text{Diff}^*(S^1) \odot C^\infty(S^1)$ by setting
\[
\langle \hat{u}, \hat{f} \rangle_\xi = \langle d_\xi R_{\hat{\xi}^{-1}}\hat{u}, d_\xi R_{\hat{\xi}^{-1}}\hat{f} \rangle_{L^2}
\]
for any $\hat{\xi} \in \hat{\mathcal{G}}$ and $\hat{u}, \hat{f} \in T_{\hat{\xi}}\hat{\mathcal{G}}$, where
\[
R_{\hat{\xi}} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}
\]
is the right translation by $\hat{\xi}$.

We shall show that the Ito equation is precisely the Euler-Arnold equation on the dual space of $\hat{G}$ associated with the $L^2$ inner product.

Given any three elements
\[
\hat{f} = (f \frac{d}{dx}, a, \alpha), \quad \hat{g} = (g \frac{d}{dx}, b, \beta) \quad \hat{u} = (u \frac{d}{dx}, v, \gamma)
\]
in $\hat{G}$.

**Lemma 1.**

\[
ad^*_f \hat{u} = \begin{pmatrix}
2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1f''' + c_2a'' \\
2f'v(x) + f(x)v'(x) - c_2f'' + 2c_3a'(x) \\
0
\end{pmatrix}
\]

**Proof.** This follows from
\[
\langle ad^*_f \hat{u}, \hat{g} \rangle_{L^2} = \langle \hat{u}, [\hat{f}, \hat{g}] \rangle_{L^2}
\]
\[
= \langle (u(x) \frac{d}{dx}, v(x), c), [(fg' - f'g)\frac{d}{dx}, fb' - ga', \omega] \rangle_{L^2}
\]
\[
= - \int_{S^1} (fg' - f'g)u(x)dx - \int_{S^1} (fb' - ga')vdx - c_1 \int_{S^1} f'(x)g''(x)dx
\]
\[
- c_2 \int_{S^1} f''(x)b(x) - g''(x)a(x))dx - 2c_3 \int_{S^1} a(x)b'(x)dx.
\]
Since $f, g, u$ are periodic functions, hence integrating by parts we obtain

\[
R.H.S. = \langle (2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1f'''(x)
\]
\[
+ c_2a''(x), f'(x)v(x) + f(x)v'(x) - c_2f''b(x) + 2c_3a'(x), 0 \rangle
\]

The coadjoint action leaves the parameter space invariant. Let us consider a hyperplane $c_1 = -1, c_2 = c_3 = 0$.

**Corollary 1.**

\[
ad^*_f \hat{u} = \begin{pmatrix}
2f'(x)u(x) + f(x)u'(x) + a'v(x) + f''' \\
2f'v(x) + f(x)v'(x) - c_2f''b(x) + 2c_3a'(x)
\end{pmatrix}
\]

The Euler-Arnold equation is the Hamiltonian flow on the coadjoint orbit in $\hat{G}^*$ [3], generated by the Hamiltonian

\[
H(\hat{u}) \equiv H(u, v) = \langle (u(x), v(x)), (u(x), v(x)) \rangle,
\]

(12)
given by
\[
\frac{d\hat{u}}{dt} = -ad^*_{\hat{u}(t)} u(t) .
\]

Let \( V \) be a vector space and assume that the Lie group \( G \) acts on the left by linear maps on \( V \), thus \( G \) acts on the left on its dual space \( V^* \) [for details, see for example, 5].

**Proposition 1.** Let \( G \circ V \) be a semidirect product space (possibly infinite dimensional), equipped with a metric \( \langle \cdot, \cdot \rangle \) which is right translation. A curve \( t \to c(t) \) in \( G \circ V \) is a geodesic of this metric if and only if \( u(t) = d_{c(t)}R_{c(t)^{-1}}\dot{c}(t) \) satisfies the Euler-Arnold equation.

Thus we prove the first part of our theorem.

### 3. \( H^1 \) Metric and Integrable Equation

Let us introduce \( H^1 \) norm on the algebra \( \hat{G} \)
\[
\langle \hat{f}, \hat{g} \rangle_{H^1} = \int_{S^1} f(x)g(x)dx + \int_{S^1} a(x)b(x)dx \int_{S^1} \partial_x f(x)\partial_x g(x)dx
\]
\[
+ \int_{S^1} \partial_x a(x)\partial_x b(x)dx + \alpha \cdot \beta ,
\]
where \( \hat{g} \) and \( \hat{f} \) are as above.

Now we compute:

**Lemma 2.** The coadjoint operator for \( H^1 \) norm is given by
\[
ad^*_{\hat{f}} \hat{u} =
\begin{pmatrix}
2f'(1-\partial_x^2)u(x) + f(x)(1-\partial_x^2)u'(x) + a'(1-\partial_x^2)v(x) - c_1f'' + c_2a'' \\
\partial_x f(x)(1-\partial_x^2)v(x) + f(x)(1-\partial_x^2)v'(x) - c_2f'' + 2c_3a'(x)
\end{pmatrix}
\]

**Proof.** From the definition it follows that
\[
\langle ad^*_{\hat{f}} \hat{u}, \hat{g} \rangle_{H^1}
\]
\[
= -\int_{S^1} (fg' - f'g)u(x)dx - \int_{S^1} (fb' - ga')vdx - c_1\int_{S^1} f'(x)g''(x)dx
\]
\[
- c_2\int_{S^1} (f''(x)b(x) - g''(x)a(x))dx - 2c_3\int_{S^1} a(x)b'(x)dx
\]
\[
- \int_{S^1} \partial_x (fg' - f'g)u(x)dx - \int_{S^1} \partial_x (fb' - ga')vdx .
\]

In the preceding section we have already computed the first five terms. After computing the last two terms by integrating by parts and using the fact that the functions \( f(x), g(x), u(x) \) and \( a(x), b(x), v(x) \) are periodic, this expression can be expressed as above.
Let us compute now the left hand side:

\[
L.H.S. = \int_{S^1} (ad^*_\xi \hat{q}) \eta \, dx + \int_{S^1} (ad^*_\xi \hat{q})' \eta' \, dx
\]
\[
= \int_{S^1} [(1 - \partial^2)ad^*_\xi \hat{q}] \eta \, dx.
\]

Thus by equating the R.H.S. and L.H.S. we obtain the above formula.

\[\square\]

**Corollary 2.**

\[
ad_f^* \hat{u} = \begin{pmatrix}
2 f'(x)(1 - \partial_x^2)u(x) + f(x)(1 - \partial_x^2)u'(x) + a'(1 - \partial_x^2)v(x) + f'''(x) \\
f'(1 - \partial_x^2)v(x) + f(x)(1 - \partial_x^2)v'(x) \\
0
\end{pmatrix}
\]

Hence by applying the proposition 1, we obtain the second part of our theorem.

**References**


S.N. Bose National Centre for Basic Sciences  
JD Block, Sector-3, Salt Lake  
Calcutta-700091, INDIA