Hana Lichardová

Saddle connections in planar systems

Archivum Mathematicum, Vol. 36 (2000), No. 5, 507--512

Persistent URL: http://dml.cz/dmlcz/107765

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 36 (2000), 507–512, CDDE 2000 issue

SADDLE CONNECTIONS IN PLANAR SYSTEMS

Hana Lichardová*

Dept. of Math., Faculty of Electrical Engineering and Information Technology SUT
Ilkovičova 3, 812 19 Bratislava, Slovakia
Email: lichardova@kmat.elf.stuba.sk

Abstract. The class of planar autonomous systems with a small parameter-dependent perturbation is considered. We derive a sufficient condition for existence of a saddle connection in such system.

AMS Subject Classification. 34C37

Keywords. Saddle connection, stable and unstable manifolds, small perturbation, Hamiltonian system.

1. Introduction

We consider systems of the form

$$\dot{x} = f(x) + \varepsilon g(x, \alpha), \quad x \in R^2, \quad \varepsilon, \alpha \in R, \tag{1}$$

where f, g are C^r, $r \geq 2$, and bounded on bounded sets, ε is a small parameter. Such systems are viewed as planar systems with a small perturbation which depends on a real parameter α. If we assume that unperturbed system (for $\varepsilon = 0$) possesses a saddle connection, then a natural question arises whether there are values of a parameter α for which a perturbed system possesses a saddle connection. There are many results related to similar questions, see for instance [1] for the problem of existence of periodic orbits in a perturbed system, or [4, §4.4], where the impact of a small time-dependent periodic perturbation on homoclinic orbit in Hamiltonian systems is studied. The paper [3] explores existence and number of periodic and homoclinic orbits, but only for a particular Hamiltonian system (whirling pendulum equation) with a special perturbation (a friction). None of the results in mentioned (and other) works has been directly applicable to our problem. To solve it, we follow a geometrical point of view as it is presented in [2].

* Research supported by grant Vega 1/6179/99

2. Assumptions and background material

We will assume that for $\varepsilon = 0$ (1) has two saddle points p_1 and p_2, which are connected by heteroclinic trajectory Γ. (The reasoning in the case of a saddle connected to itself by a homoclinic loop is very similar). More precisely, one branch, say Γ^u, of the global unstable manifold W^u of p_1 coincides with one branch, say Γ^s of the global stable manifold W^s of p_2, and they form a saddle connection Γ (see Fig. 1a).

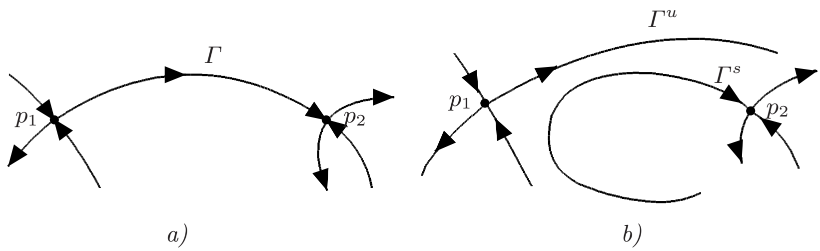

Fig. 1. The phase portrait of $\dot{x} = f(x) + \varepsilon g(x, \alpha)$ for a) $\varepsilon = 0$, b) $\varepsilon \neq 0$.

This situation is not resistent to perturbations – in general, any perturbation will break the saddle connection, although the local phase portraits will not change under a small perturbation (see Fig. 1b). Particularly, the following facts are well-known for (1) with $\varepsilon \neq 0$ (for details we refer the reader to [2, §4.5] and the references given there):

F1 For each ε sufficiently small, (1) has two unique saddles $p_1^\varepsilon = p_1^\varepsilon + \mathcal{O}(\varepsilon)$, $p_2^\varepsilon = p_1^\varepsilon + \mathcal{O}(\varepsilon)$. This is a straightforward application of the implicit function theorem, since Jacobi matrices $Df(p_1)$, $Df(p_2)$ are invertible (they have nonzero real eigenvalues).

F2 Perturbed local stable and unstable manifolds of the saddles p_1^ε, p_2^ε'are C^r-close to unperturbed local stable and unstable manifolds of the saddles p_1, p_2. This fact follows from invariant manifold theory.

F3 If we denote by $\gamma(t)$ a solution of the unperturbed system lying in Γ, by $\gamma^u(t)$ and $\gamma^s(t)$ solutions of the perturbed system lying in Γ_ε^u and Γ_ε^s (branches of W_ε^u and W_ε^s corresponding to Γ^u and Γ^s), the following expressions holds, with uniform validity in the indicated intervals:

$$(2) \qquad \begin{aligned} \gamma^s(t) &= \gamma(t) + \varepsilon\gamma_1^s(t) + \mathcal{O}(\varepsilon^2), \quad t \in [0, \infty), \\ \gamma^u(t) &= \gamma(t) + \varepsilon\gamma_1^u(t) + \mathcal{O}(\varepsilon^2), \quad t \in (-\infty, 0]. \end{aligned}$$

Here $\gamma_1^s(t)$ and $\gamma_1^u(t)$ are solutions of the first variational equations

$$(3) \qquad \dot{\gamma}_1^{s,u}(t) = Df(\gamma(t))\gamma_1^{s,u}(t) + g(\gamma(t), \alpha).$$

This fact represents both local and global dynamics – near a saddle point (infinite time interval) it is governed by exponential attraction and repulsion, while away from a saddle (finite time interval) the closeness of solutions may be derived thanks to Gronwall's inequality.

In what follows, we will look for values of parameter α for which the saddle connection persists. The main idea is to measure, in some sense, the distance between perturbed branches Γ_ε^u and Γ_ε^s of the global manifolds W_ε^u and W_ε^s.

3. The distance function

Let $p \in \Gamma$ be a nonsingular point ($f(p) \neq 0$), and $p^u \in \Gamma_\varepsilon^u$, $p^s \in \Gamma_\varepsilon^s$ are lying on the normal $f^{\perp(p)}$ to Γ at p (Fig. 2). Then we define the oriented distance between Γ_ε^u and Γ_ε^s at the point p as

$$d(\varepsilon, \alpha) = \frac{f(p) \wedge (p^u - p^s)}{|f(p)|},$$

where $a \wedge b = a^\perp \cdot b$ is the wedge product.

We denote $\gamma(t)$, $\gamma^s(t)$ and $\gamma^u(t)$ solutions lying in Γ, Γ_ε^s and Γ_ε^u for which

$$\gamma(0) = p, \quad \gamma^s(0) = p^s, \quad \gamma^u(0) = p^u. \tag{4}$$

Using (2) and (4), we can write

$$d(\varepsilon, \alpha) = \varepsilon \frac{f(\gamma(0)) \wedge (\gamma_1^u(0) - \gamma_1^s(0))}{|f(\gamma(0))|} + \mathcal{O}(\varepsilon^2).$$

Now we define the time dependent distance function

$$\Delta(t) = f(\gamma(t)) \wedge (\gamma_1^u(t) - \gamma_1^s(t))$$

which may be written as $\Delta(t) = \Delta^u(t) - \Delta^s(t)$ with $\Delta^{s,u}(t) = f(\gamma(t)) \wedge \gamma_1^{s,u}(t)$. Note that

$$d(\varepsilon, \alpha) = \varepsilon \frac{\Delta(0)}{|f(\gamma(0))|} + \mathcal{O}(\varepsilon^2).$$

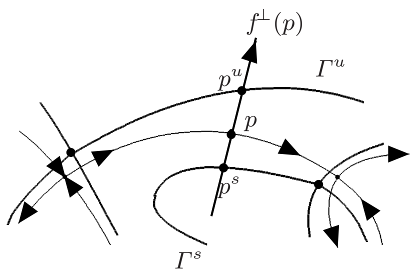

Fig. 2. Definition of the distance function.

The derivative of $\Delta^{s,u}(t)$ with respect to time is

$$\dot{\Delta}^{s,u}(t) = Df(\gamma(t))\dot{\gamma}(t) \wedge \gamma_1^{s,u}(t) + f(\gamma(t)) \wedge \dot{\gamma}_1^{s,u}(t).$$

Using (3) and the fact that $\dot{\gamma}(t) = f(\gamma(t))$, we obtain, after some matrix calculations,

$$\dot{\Delta}^{s,u}(t) = \mathrm{Tr}\ (Df(\gamma(t)))\, \Delta^{s,u}(t) + f(\gamma(t)) \wedge g(\gamma(t), \alpha).$$

Integrating the last equation from 0 to ∞ for Δ^s and from $-\infty$ to 0 for Δ^u yields

$$\Delta^s(\infty) - \Delta^s(0) = \int_0^\infty f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{-\int_0^t \mathrm{Tr}\ (Df(\gamma(s)))\, \mathrm{d}s} \mathrm{d}t,$$

$$\Delta^u(0) - \Delta^u(-\infty) = \int_{-\infty}^0 f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{\int_t^0 \mathrm{Tr}\ (Df(\gamma(s)))\, \mathrm{d}s} \mathrm{d}t.$$

Since

$$\Delta^s(\infty) = \lim_{t\to\infty} f(\gamma(t)) \wedge \gamma_1^s(t),$$

where $\gamma_1^s(t)$ is bounded and $\lim_{t\to\infty} f(\gamma(t)) = f(p_2) = 0$, we have $\Delta^s(\infty) = 0$. Similarly $\Delta^u(-\infty) = 0$. Then

$$\Delta^s(0) = \int_0^\infty f(\gamma(t)) \wedge g(\gamma(t), \alpha) e^{-\int_0^t \mathrm{Tr}\ (Df(\gamma(s)))\, \mathrm{d}s} \mathrm{d}t.$$

In the case when the unperturbed system is Hamiltonian, i.e. $f = \left(\frac{\partial H}{\partial x_2}, -\frac{\partial H}{\partial x_1}\right)$ for some differentiable function $H(x_1, x_2)$, we have $\mathrm{Tr}\,(Df) \equiv 0$, and

$$\Delta(0) = \int_{-\infty}^{\infty} f(\gamma(t)) \wedge g(\gamma(t), \alpha) dt,$$

which is the homoclinic Melnikov function [2, p. 187].

In the next, we will use more suitable notation $\Delta(0) = M(\alpha)$, which takes into account the fact that $\Delta(0)$ depends on α. Thus

$$d(\varepsilon, \alpha) = \varepsilon \frac{M(\alpha)}{|f(p)|} + \mathcal{O}(\varepsilon^2). \tag{5}$$

Now we are ready to state and prove the main result:

Theorem 1. *Let there exist α_0 such that $M(\alpha_0) = 0$, $M'(\alpha_0) \neq 0$. Then for each ε sufficiently small there exists $\alpha(\varepsilon) = \alpha_0 + \mathcal{O}(\varepsilon)$ such that the perturbed system*

$$\dot{x} = f(x) + \varepsilon g(x, \alpha(\varepsilon))$$

possesses a saddle connection, which is C^r-close to the saddle connection of the unperturbed system.

Proof. We rewrite (5) in the form $d(\varepsilon,\alpha) = \varepsilon\overline{d}(\varepsilon,\alpha)$, where

$$\overline{d}(\varepsilon,\alpha) = \frac{M(\alpha)}{|f(p)|} + \mathcal{O}(\varepsilon).$$

Then, for $\varepsilon \neq 0$, d vanishes if and only if $\overline{d}$ vanishes. For α_0 with indicated properties we obtain

$$\overline{d}(0,\alpha_0) = 0, \qquad \frac{\partial\overline{d}}{\partial\alpha}(0,\alpha_0) \neq 0.$$

The implicit function theorem ensures the existence of a smooth curve of points $(\varepsilon,\alpha(\varepsilon))$ passing throw $(0,\alpha(0))$, $\alpha(0) = \alpha_0$, with a property

$$\overline{d}(\varepsilon,\alpha(\varepsilon)) = 0.$$

It means that the oriented distance between Γ^u_ε and Γ^s_ε at the point p is zero, which implies, thanks to the uniqueness theorem, that they coincide, forming a saddle connection. The C^r-closeness is ensured by F3.

4. Example

We will seek parameter α_0 for which there exists a smooth curve of parameters $\alpha(\varepsilon)$ with the property: the planar system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x + \varepsilon y(\cos x + \alpha(\varepsilon)) \end{aligned} \tag{6}$$

has a saddle connection that is C^r-close to the upper saddle connection of the planar pendulum equation, i.e. the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin x. \end{aligned} \tag{7}$$

To obtain the value of α_0, we will compute $M(\alpha)$ for (6). First, we recall that the planar pendulum equation (7) is a Hamiltonian system with the energy

$$H(x,y) = \frac{y^2}{2} - \cos x + 1.$$

Saddles $-\pi$, π are connected by two heteroclinic orbits

$$y = \pm\sqrt{2(\cos x + 1)}$$

(upper and lower saddle connections) corresponding to the energy level $h = 2$. Then

$$M(\alpha) = \int_{-\infty}^{\infty} y^2(t)(\cos x(t) + \alpha)\mathrm{d}t.$$

Using the fact that $y\mathrm{d}t = \mathrm{d}x$, and trigonometrical identity $\cos x + 1 = 2\cos^2\dfrac{x}{2}$, we obtain that along the upper saddle connection

$$M(\alpha) = \int_{-\pi}^{\pi} y(\cos x + \alpha)\mathrm{d}x = 8(\alpha + \frac{1}{3}).$$

Consequently, if we denote $\alpha_0 = -\dfrac{1}{3}$, then

$$M(\alpha_0) = 0, \qquad M'(\alpha_0) \neq 0.$$

By Theorem 1, for each ε sufficiently small there exists $\alpha(\varepsilon) = -\frac{1}{3} + \mathcal{O}(\varepsilon)$ such that (6) has an upper saddle connection. Moreover, from the definition of $d(\varepsilon, \alpha)$ we can deduce that for $\alpha > \alpha(\varepsilon)$ the unstable manifold of $[-\pi, 0]$ is lying above the stable manifold of $[\pi, 0]$, and reversely for $\alpha < \alpha(\varepsilon)$ (see Fig. 3, where the situation is depicted for two values of ε). The similar result may be obtained for the lower saddle connection.

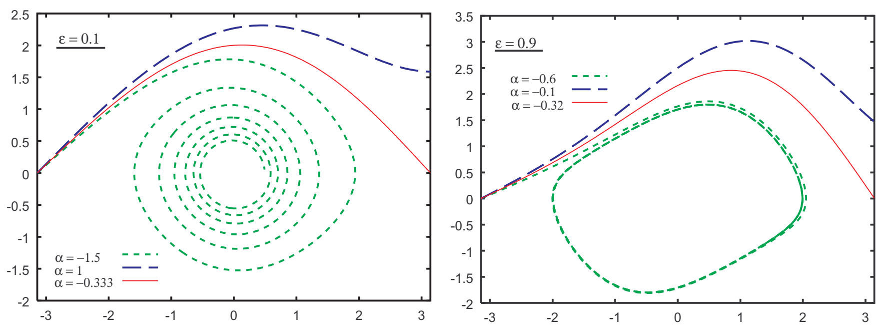

Fig. 3. Phase portraits of (6).

References

1. Chicone C. *On bifurcation of limit cycles from centers.* Lecture Notes in Math., **1991(1455)**, 20–43, 1991.
2. Guckenheimer J. and Holmes P. J. *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* Springer Verlag, New York, Heidelberg, Berlin, 1983.
3. Lichardová H. *Limit cycles in the equation of whirling pendulum with autonomous perturbation.* Applications of Mathematics, **44(4)** (1999), 271–288.
4. Wiggins S. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer Verlag, New York, Heidelberg, Berlin, 1990.