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NATURAL VECTOR FIELDS AND 2-Vектор FIELDS ON THE TANGENT BUNDLE OF A PSEUDO-RIEMANNIAN MANIFOLD

JOSEF JANÝŠKA

Abstract. Let $M$ be a differentiable manifold with a pseudo-Riemannian metric $g$ and a linear symmetric connection $K$. We classify all natural (in the sense of [4]) 0-order vector fields and 2-vector fields on $TM$ generated by $g$ and $K$. We get that all natural vector fields are of the form

$$E(u) = \alpha(h(u)) u^H + \beta(h(u)) u^V,$$

where $u^V$ is the vertical lift of $u \in T_x M$, $u^H$ is the horizontal lift of $u$ with respect to $K$, $h(u) = 1/2g(u,u)$ and $\alpha, \beta$ are smooth real functions defined on $\mathbb{R}$. All natural 2-vector fields are of the form

$$\Lambda(u) = \gamma_1(h(u)) \Lambda(g,K) + \gamma_2(h(u)) u^H \wedge u^V,$$

where $\gamma_1, \gamma_2$ are smooth real functions defined on $\mathbb{R}$ and $\Lambda(g,K)$ is the canonical 2-vector field induced by $g$ and $K$. Conditions for $(E,\Lambda)$ to define a Jacobi or a Poisson structure on $TM$ are discussed.

Introduction

In this paper $M$ is a differentiable manifold with a of pseudo-Riemannian metric $g$. Let $(x^i)$ be a typical local chart on $M$, then $(\partial_i)$ and $(d^i)$ denote the canonical local bases of modules of vector fields and forms on $M$. In general relativistic theories $\dim M = 4$ and $g$ is a Lorentz metric, but it is not relevant for our purposes; our considerations are correct for non-orientable manifolds if $\dim M \geq 2$ and for orientable manifolds if $\dim M \geq 4$.

The isomorphism $T^*M \rightarrow TM$ given by the metric tensor will be as usual denoted by $\sharp$ and its inverse by $\flat$.

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We consider the tangent bundle $p_M: TM \to M$ of $M$. The natural fibred coordinates on $TM$ are denoted by $(x^i, \dot{x}^i)$ and the canonical local bases of modules of vector fields and forms are denoted by $(\partial_i, \dot{\partial}_i)$ and $(d^i, d\dot{i})$.

The canonical natural function (kinetic energy) on $TM$ will be denoted by $h(u) = \frac{1}{2}g(u, u)$, $u \in TM$.

We assume a linear symmetric connection (gravitational field) $K$ on $M$.

We use the term “natural operator” in the sense of [4, 6, 11]. Namely, a natural operator is defined to be a system of local operators $A_M: C^\infty(FM) \to C^\infty(GM)$, such that $A_N(f^r_s) = f^r_{cs}A_M(s)$ for any section $(s: M \to FM) \in C^\infty(FM)$ and any (local) diffeomorphism $f: M \to N$, where $F, G$ are two natural bundles, [9].

A natural operator is said to be of order $r$ if, for all sections $s, q \in C^\infty(FM)$ and every point $x \in M$, the condition $f^r_s = f^r_{q}q$ implies $A_M s(x) = A_M q(x)$. Then we have the induced natural transformation $A_M: J^r FM \to GM$ such that $A_M(s) = A_M(f^r_s)$, for all $s \in C^\infty(FM)$. It is well known, that the correspondence between natural operators of order $r$ and the induced natural transformations is bijective. In this paper by natural operators we mean the corresponding natural transformations. Briefly speaking, a natural operator is a fibred manifold mapping which is invariant with respect to local diffeomorphisms of the underlying manifold.

In [1, 2] we have classified all natural 2-form fields on $TM$ generated by $g$ and $K$ and we have found conditions for such 2-form fields to be symplectic. The aim of this paper is to classify all natural vector fields and 2-vector fields on $TM$ generated by $g$ and $K$. Finally, in Sections 5 and 6, we recall, [3], conditions under which natural vector fields and 2-vector fields define a Poisson or a Jacobi structure on $TM$. Evidently the conditions for the natural nondegenerate Poisson structure have to be equivalent with conditions for the natural symplectic structure we have found in [1, 2].

All manifolds and mappings are assumed to be smooth.

1. Schouten-Nijenhuis bracket

Let $\mathcal{V}^k(M) \quad (\mathcal{V}^0(M) = C^\infty(M))$ denote the space of $k$-vector fields on a differentiable manifold $M$ and $\mathcal{V}(M) = (\oplus_{k=0}^p \mathcal{V}^k(M), \wedge)$ be the contravariant Grassmann algebra of $M$. Let us recall that the Schouten-Nijenhuis bracket is a (natural) bilinear map

$$(P, Q) \in \mathcal{V}^p(M) \times \mathcal{V}^q(M) \to [P, Q] \in \mathcal{V}^{p+q-1}(M)$$

satisfying the following properties:

1. $[P, Q] = (-1)^{pq}[Q, P]$;
2. $(-1)^{(p-1)}(s-1)[P, [Q, S]] + (-1)^{(q-1)(p-1)}[Q, [S, P]]$
   $\quad + (-1)^{(s-1)(q-1)}[S, [P, Q]] = 0$;
3. $[P, Q \wedge S] = [P, Q] \wedge S + (-1)^{pq-q}Q \wedge [P, S]$;
4. $[X, Q] = \mathcal{L}_X Q$, where $\mathcal{L}_X$ is the Lie derivative;

for any $P \in \mathcal{V}^p(M)$, $Q \in \mathcal{V}^q(M)$, $S \in \mathcal{V}^r(M)$, $X \in \mathcal{V}^1(M)$. 

For any function \( f \in \mathcal{V}^0(M) = C^\infty(M) \) the equality (3) implies
\[
[P, fQ] = [P, f] \wedge Q + f[P, Q],
\]
where \([P, f]\) is defined by
\[
[P, f](\rho_2, \ldots, \rho_p) = P(df, \rho_2, \ldots, \rho_p)
\]
for any 1-form fields \( \rho_i \) on \( M \).

The Schouten-Nijenhuis bracket can be characterized by
\[
i([P, Q])^\beta = (-1)^{q(p+1)}i(P)d[i(Q)\beta] + (-1)^pi(Q)d[i(P)\beta] - i(P \wedge Q)d\beta
\]
for any \((p + q - 1)\)-form field \( \beta \).

A 2-vector field \( \Lambda \) defines on \( M \) a Poisson structure, \([7, 12]\), if
\[
[\Lambda, \Lambda] = 0.
\]
A pair \((E, \Lambda)\), where \( E \) is a vector field and \( \Lambda \) is a 2-vector field, defines on \( M \) a Jacobi structure, \([8]\), if
\[
[E, \Lambda] = 0, \quad [\Lambda, \Lambda] = 2E \wedge \Lambda.
\]

2. Natural vector fields on \( TM \)

We can define two canonical natural vector fields on \( TM \). The first one is the Liouville vector field \( \ell(u) \), which can be considered as the vertical lift \( u^V \) of \( u \in T_uM \). In coordinates
\[
u^V = \ell(u) = \dot{x}^i \partial_i.
\]
A linear connection \( K \) on \( TM \) is a linear \( TTM \)-valued 1-form
\[
K: TM \rightarrow T^*M \otimes TTM
\]
with coordinate expression
\[
K = d^s \otimes (\partial_i + K_i^j_k \dot{x}^k \partial_j), \quad K_i^j_k \in C^\infty(M).
\]
The space of linear connections is a natural bundle of 2nd order and we shall denote it by \( CM \), by \( C_TM \) we shall denote the subbundle of torsion-free linear connections on \( M \). If we consider \( K \) as the mapping \( K: TM \times_M TM \rightarrow TTM \), called the horizontal lift, we can define the horizontal lift of \( u \) as the vector field \( u^H = K(u, u) \), i.e. in coordinates
\[
u^H = \dot{x}^i \partial_i + K_i^j_k \dot{x}^r \dot{x}^s \partial_i.
\]
The vertical and the horizontal lifts of \( u \) are natural 0-order (with respect to the connection \( K \)) vector fields on \( TM \), i.e. they are natural operators from \( TM \times CM \) into \( TTM \) projectable (with respect to the projections \( p_1: TM \times CM \rightarrow TM \) and \( p_{TM}: TTM \rightarrow TM \) over the identity of \( TM \). As a consequence of the results of \([10]\) we get that all 0-order natural vector fields on \( TM \) given by \( K \) are linear combination (with real coefficients) of the vertical and the horizontal lifts. In what follows we shall consider natural 0-order vector fields induced by a pseudo-Riemannian metric \( g \) and a torsion-free linear connection, i.e. we shall
classify all 0-order natural operators from $TM \times_M \mathcal{R} \otimes^2 T^*M \times_M C_T^M$ into $TTM$ projectable over the identity of $TM$.

**Theorem 2.1.** Let $M$ be a differentiable manifold with a pseudo-Riemannian metric $g$ and a linear symmetric connection $K$. Then all natural 0-order vector fields on $TM$ are of the form

$$E(u) = \alpha(h(u)) u^H + \beta(h(u)) u^V,$$

where $\alpha, \beta$ are smooth real functions defined on $\mathbb{R}$.

**Proof.** According to the general theory of natural operators, [4, 6], all natural 0-order operators from $TM \times_M \mathcal{R} \otimes^2 T^*M \times_M C_T^M$ into $TTM$ are given by invariant mappings from the standard fibre $Q$ of the functor $T \times_M \mathcal{R} \otimes^2 T^* \times C_T$ into the standard fibre $S$ of the functor $TT$. To classify these invariant mappings we shall use the infinitesimal method of [6].

The standard fibre $Q = \mathbb{R}^n \times \mathcal{R}(\mathbb{R}^n \otimes \mathbb{R}^n) \times \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ with coordinates $(\dot{x}^i, g_{ij}, K^i_{jk})$, $g_{ij} = g_{ji}$, $\det(g_{ij}) \neq 0$, $K^i_{jk} = K^j_{ki}$, and the action of the 2nd order differential group $G^2_n = \text{inv.}^0(\mathbb{R}^n, \mathbb{R}^n)_0$ given in coordinates by

$$\bar{x}^i = a^i_j \dot{x}^j, \quad \bar{g}_{ij} = \bar{a}^i_j \bar{a}^j_i g_{pq}, \quad \bar{K}^i_{jk} = a^i_r K^r_{pq} \bar{a}^p_j \bar{a}^q_i - a^i_r \bar{a}^r_i \bar{a}^j_r,$$

where $(a^i_j, a^i_{jk})$ are the canonical coordinates on $G^2_n$ and tilde denotes the inverse element.

The fundamental vector fields on $Q$ relative to this action are

$$\zeta^q_p(Q) = \dot{x}^q \frac{\partial}{\partial \dot{x}^p} - 2g_{ip} \frac{\partial}{\partial g_{ip}} + (\delta^q_p K^i_{jk} - \delta^q_j K^i_{kp} - \delta^q_k K^i_{jp}) \frac{\partial}{\partial K^i_{jk}},$$

$$\zeta^r_p(Q) = \frac{1}{2} \left( \frac{\partial}{\partial K^r_{pq}} + \frac{\partial}{\partial K^r_{qp}} \right) = \frac{\partial}{\partial K^r_{pq}}.$$

The standard fibre $S = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(\dot{x}^i, \xi^i, \Xi^i)$ and the action of the group $G^2_n$ given in coordinates by

$$\bar{x}^i = a^i_p \dot{x}^p, \quad \bar{\xi}^i = a^i_p \xi^p, \quad \bar{\Xi}^i = a^i_p \Xi^p + a^i_p \xi^p \xi^q.$$

The fundamental vector fields on $S$ relative to this action are

$$\zeta^q_p(S) = \dot{x}^q \frac{\partial}{\partial \dot{x}^p} + \xi^q \frac{\partial}{\partial \xi^p} + \Xi^q \frac{\partial}{\partial \Xi^p},$$

$$\zeta^r_p(S) = \frac{1}{2} \left( \xi^q \xi^p + \dot{x}^q \dot{x}^p \right) \frac{\partial}{\partial \Xi^p}.$$

A mapping $F : Q \to S$ is $G^2_n$-invariant if and only if the corresponding fundamental vector fields are $F$-related. Let $F = (\dot{x}^i(\dot{x}^r, g_{pq}, K^r_{pq}), \xi^i(\dot{x}^r, g_{pq}, K^r_{pq}), \Xi^i(\dot{x}^r, g_{pq}, K^r_{pq}))$ be a coordinate expression of $F$. The assumption that the operator is projectable over the identity of $TM$ implies $\dot{x}^i = \dot{\bar{x}}^i$. Further

$$\dot{x}^q \frac{\partial \dot{\xi}^i}{\partial \dot{x}^p} - 2g_{ip} \frac{\partial \dot{\xi}^i}{\partial g_{ip}} + (\delta^q_p K^i_{jk} - \delta^q_j K^i_{kp} - \delta^q_k K^i_{jp}) \frac{\partial \dot{\xi}^i}{\partial K^i_{jk}} = \delta^i_p \xi^q,$$

$$\dot{x}^q \frac{\partial \dot{\Xi}^i}{\partial \dot{x}^p} - 2g_{ip} \frac{\partial \dot{\Xi}^i}{\partial g_{ip}} + (\delta^q_p K^i_{jk} - \delta^q_j K^i_{kp} - \delta^q_k K^i_{jp}) \frac{\partial \dot{\Xi}^i}{\partial K^i_{jk}} = \delta^i_p \Xi^q.$$
and
\( \frac{\partial \xi^i}{\partial K^p_q r} = 0 , \) \hspace{1cm} (2.12) \\
\( \frac{\partial \Xi^i}{\partial K^p_q r} = \frac{1}{2} \delta^i_p (\xi^q \dot{x}^r + \dot{x}^q \xi^r) . \) \hspace{1cm} (2.13)

Putting (2.12) into (2.10) we get
\( \dot{x}^q \frac{\partial \xi^i}{\partial \dot{x}^p} - 2g_{ip} \frac{\partial \xi^i}{\partial g_{iq}} = \delta^i_p \xi^q , \) \hspace{1cm} (2.14)

which implies, that \( \xi^i \) is a \( G^1_n \)-invariant mapping from \( \mathbb{R}^n \times \text{reg}(\mathbb{R}^n \odot \mathbb{R}^n) \) into \( \mathbb{R}^n \) which is the \( G^1_n \)-invariant mapping corresponding to 0-order natural operator from \( TM \times \text{reg} \odot^2 T^*M \) to \( TM \). From the classical theory of differential invariants it is known that all such mappings are of the form
\( \xi^i = \alpha(h(u)) \dot{x}^i , \) \hspace{1cm} (2.15)

where \( \alpha \) is a smooth real function defined on \( \mathbb{R} \).

The right hand side of (2.13) we can rewrite in the form
\( \frac{1}{2} \delta^i_p (\xi^q \dot{x}^r + \dot{x}^q \xi^r) = \frac{1}{2} \delta^i_p (\delta^q_r \delta^i_l + \delta^q_l \delta^i_r) \xi^m \dot{x}^l = \frac{\partial K^i_m}{\partial K^q_p r} \xi^m \dot{x}^l , \) \hspace{1cm} (2.16)

which implies that
\( \Xi^i = K^i_p q \xi^p \dot{x}^q + Z^i , \) \hspace{1cm} (2.17)

where \( Z^i \) is a function independent on \( K_j k \). Putting (2.16) into (2.11) we get that \( Z^i \) has to satisfy (2.14), i.e.
\( Z^i = \beta(h(u)) \dot{x}^i , \) \hspace{1cm} (2.18)

where \( \beta \) is a smooth real function defined on \( \mathbb{R} \).

Now, putting (2.15) and (2.17) into (2.16), we have
\( \xi^i = \alpha(h(u)) \dot{x}^i , \quad \Xi^i = \alpha(h(u)) K^i_p q \dot{x}^p \dot{x}^q + \beta(h(u)) \dot{x}^i , \) \hspace{1cm} (2.19)

and it is easy to see that the natural operator corresponding to (2.18) is the vector field (2.3).

**Lemma 2.2.** We have:
1. \( [u^V, u^H] = u^H ; \)
2. \( [u^V, u^V \wedge u^H] = u^V \wedge u^H ; \)
3. \( [u^H, u^V \wedge u^H] = 0 ; \)
4. \( [u^V \wedge u^H, u^V \wedge u^H] = 0 . \)

**Proof.** It is easy to prove the first equality by direct coordinate calculation by using (2.1) and (2.2). The others equalities follows from the properties of the Schouten-Nijenhuis bracket.

**Lemma 2.3.** Let \( \gamma \) be a smooth real function defined on \( \mathbb{R} \). Then we have
\( [u^V, \gamma(h(u))] = 2h(u) \dot{\gamma}(h(u)) , \) \hspace{1cm} (2.19)
\( [u^H, \gamma(h(u))] = \frac{1}{2} \dot{\gamma}(h(u))(\nabla g)(u,u)(u) , \) \hspace{1cm} (2.20)
where \( \dot{\gamma} = \frac{d\gamma}{dt} \) and \( \nabla \) is the covariant differentiation with respect to \( K \).

**Proof.** For any vector field \( E \) we have \( [E, \gamma(h)] = \langle E, d\gamma(h) \rangle = \dot{\gamma}(h)\langle E, dh \rangle \). Lemma 2.3 now follows from (2.1), (2.2) and \( dh = \frac{1}{2}g_{pq}i \dot{x}^p \dot{x}^q d^i + g_{pi} \dot{x}^p d^i \). \( \square \)

3. Canonical 2-vector field generated by \( g \) and \( K \)

Let us denote by \( \vartheta \) the \( VTM \)-valued 1-form on \( M \) given by the vertical lift, i.e.

\[
\vartheta : M \rightarrow T^*M \otimes VTM , \quad \vartheta = d^i \otimes \dot{\vartheta}_i .
\]

The metric \( g \) and the connection \( K \) induce naturally a 2-vector

\[
\Lambda(g, K) = K\bar{\vartheta} : TM \rightarrow \wedge^2 T(TM) ,
\]

where \( \bar{\wedge} \) denotes the wedge product followed by the contraction through the inverse metric \( \bar{g} \). In coordinates

\[
\Lambda(g, K) = g^{ij}(\partial_i + K_i^m \dot{x}^k \dot{\vartheta}_m) \wedge \dot{\vartheta}_j .
\]

**Remark 3.1.** The canonical 2-vector field \( \Lambda(g, K) \) can be characterized by

\[
\Lambda(g, K)(\rho^V, \sigma^V) = 0 , \quad \Lambda(g, K)(\rho^H, \sigma^H) = \bar{g}(\rho, \sigma) ,
\]

\[
\Lambda(g, K)(\rho^H, \sigma^V) = -\bar{g}(\rho, \sigma) , \quad \Lambda(g, K)(\rho^V, \sigma^H) = 0 ,
\]

where \( \rho, \sigma \) are 1-form fields on \( M \), \( \rho^V, \sigma^V \) are their vertical lifts (pullbacks) and \( \rho^H, \sigma^H \) are horizontal lifts with respect to \( K \).

The metric \( g \) can be considered to be a \( T^*M \)-valued 1-form on \( M \) which will be denoted by \( \bar{g} \) to distinguish it from the metric. Then, [4], we define the **covariant exterior differential** of \( \bar{g} \) as a \( T^*M \)-valued 2-form field \( d_K \bar{g} \) defined for any vector fields \( X_1, X_2, X_3 \) by

\[
d_K \bar{g}(X_1, X_2)(X_3) = (\nabla_{X_1} \bar{g}(X_2) - \nabla_{X_2} \bar{g}(X_1) - \bar{g}([X_1, X_2]))(X_3) .
\]

Then

\[
d_K^2 \bar{g} = R \wedge \bar{g} ,
\]

where \( R = R^i_{jkr} \dot{x}^r \partial_i \otimes d^j \wedge d^k \) is the curvature tensor field of \( K \). The \( T^*M \)-valued 3-form \( R \wedge \bar{g} \) is defined by

\[
(R \wedge \bar{g})(X_1, X_2, X_3)(X_4) = \frac{1}{2!} \sum_{\sigma} |\sigma| g(X_{\sigma(3)}, R(X_{\sigma(1)}, X_{\sigma(2)}))(X_4) ,
\]

where \( \sigma \) is a permutation and \( |\sigma| \) is its sign. In coordinates

\[
d_K^2 \bar{g} = R^p_{ijr} g_{pk} d^r \otimes d^i \wedge d^j \wedge d^k .
\]

**Lemma 3.2.** Let \( X_1, X_2, X_3 \) be three vector fields on \( M \). Then

\[
d_K \bar{g}(X_1, X_2)(X_3) = 0
\]

if and only if

\[
(\nabla_{X_1} g)(X_2, X_3) = (\nabla_{X_2} g)(X_1, X_3) ,
\]

i.e. if and only if \( \nabla g \) is a section \( \nabla g : M \rightarrow \wedge^3 T^*M \).
Proof. We have
\[ \nabla X_1 X_2 - \nabla X_2 X_1 = [X_1, X_2] \]
and
\[ \nabla X_1 g(X_2, X_3) - g(X_2, \nabla X_1 X_3) = \nabla X_1 \bar{g}(X_2)(X_3). \]

Then
\[ (\nabla X_1 g)(X_2, X_3) - (\nabla X_2 g)(X_1, X_3) = d_K \bar{g}(X_1, X_2)(X_3) \]
implies Lemma 3.2.

Corollary 3.3. The condition (A) is satisfied if and only if
\[ X_1 g(X_2, X_3) - X_2 g(X_1, X_3) \]
for any three vector fields $X_1, X_2, X_3$ on $M$.

Theorem 3.4. $\Lambda(g, K)$ defines a Poisson structure if and only if the condition (A) is satisfied.

Proof. We have from (1.1)
\[ [\Lambda(g, K), \Lambda(g, K)] = g^{ip} g^{rj} g^{mk} (g_{pm,r} + g_{ms} K_s^p r) \partial_i \wedge \dot{\partial}_j \wedge \dot{\partial}_k \]
\[ + (R^{ijk}_{r} + R^{jki}_{r} + R^{kij}_{r}) \hat{x}^r \dot{\partial}_i \wedge \dot{\partial}_j \wedge \dot{\partial}_k. \]

On the other hand
\[ d_K \bar{g} = (g_{ij,k} + g_{js} K_i^s k) d^i \wedge d^j \wedge d^k. \]
The condition $d_K \bar{g} = 0$ imply $d^2_K \bar{g} = 0$ which is from (3.4) equivalent to
\[ R_{ijk} + R_{jki} + R_{kij} = 0, \]
i.e. by increasing indices
\[ R_{ijk} + R_{jki} + R_{kij} = 0, \]
which implies Theorem 3.4.

Lemma 3.5. Let $\gamma$ be a smooth real function defined on $\mathbb{R}$. Then we have:
(1) $[\Lambda(g, K), \gamma(h(u))] = \gamma(h(u)) \left( -u^H + \frac{1}{2} \left( ((\nabla g)(u, u))^2 \right)^V \right),$
(2) $[\Lambda(g, K), u^V] = -\Lambda(g, K).$

Proof. It is easy to prove it in coordinates.

Lemma 3.6.
\[ (((\nabla g)(u, u))^2)^V \wedge u^H = 0 \]
if and only if
\[ \nabla g = 0. \]

Proof. In coordinates we have
\[ (((\nabla g)(u, u))^2)^V \wedge u^H = (\nabla_m g_{rs}) \hat{x}^r \hat{x}^s g^{mi} \dot{\partial}_i \wedge (\hat{x}^j \partial_j + K^j_{pq} \hat{x}^p \hat{x}^q \partial_j), \]
which vanishes if and only if $\nabla_m g_{rs} = 0.$
Lemma 3.7. The following relations are equivalent:

(B) $g(X_1, X_3)(\nabla_{X_2} g)(X_4, X_5) = g(X_2, X_3)(\nabla_{X_1} g)(X_4, X_5)$ for any vector fields $X_i, i = 1, \ldots, 5$.

(1) $((\nabla g)(u, u))^V \land \Lambda(g, K) = 0$;

(2) $((\nabla g)(u, u))^V \land u^V = 0$.

Proof. The condition (B) in coordinates reads as

$$g_{ij} \nabla_k g_{rs} - g_{kj} \nabla_i g_{rs} = 0.$$  

(B) $\iff$ (1) In coordinates we have

$$\Lambda((\nabla g)(u, u))^V \land \Lambda(g, K) = -g^{pk} g^{mj}_{\nabla m g_{rs}} \dot{x}^r \dot{x}^s (\delta^i_p \partial_i \land \partial_j \land \partial_k + K^i_{pq} \dot{x}^q \partial_i \land \partial_j \land \partial_k).$$

Now

$$(g^{pk} g^{mj} - g^{pj} g^{mk})(\nabla m g_{rs}) = (g^{pk} g^{mj} g_{ab} g^{bp} - g^{pj} g^{mk} g_{ab} g^{bp})(\nabla m g_{rs}) = g^{pk} g^{mj} g^{bp}(g_{ab} \nabla_m g_{rs} - g_{mb} \nabla_a g_{rs})$$

which implies the first equivalence.

(B) $\iff$ (2) In coordinates we have

$$((\nabla g)(u, u))^V \land u^V = g^{mi} \dot{x}^j (\nabla_m g_{rs}) \dot{x}^r \dot{x}^s \partial_i \land \partial_j.$$  

The equivalence now follows from

$$(g^{mj} \dot{x}^k - g^{mk} \dot{x}^j)(\nabla m g_{rs}) = g^{jm} g^{kp} \dot{x}^q (g_{pq} \nabla_m g_{rs} - g_{mq} \nabla_p g_{rs}).$$ \hfill \Box

Lemma 3.8. We have

$$[\Lambda(g, K), u^H] = 0 \iff \nabla g = 0.$$  

Proof. In coordinates

$$[\Lambda(g, K), u^H] = \dot{x}^r \nabla_r g^{ij} \partial_i \land \partial_j + (R^{ij}_{\,rs} + K^{ij}_{\,rs} \nabla s g^{mj}) \dot{x}^r \dot{x}^s \partial_i \land \partial_j.$$  

If $K$ is the metric connection, i.e. $\nabla g = 0$ then $g(R(X_1, X_2)(X_3), X_3) = 0$ which in coordinates reads as

$$R_{rijs} \dot{x}^r \dot{x}^s = 0.$$  

Then, by using the cyclic permutations of the first three indices, we have

$$0 = R_{rijs} \dot{x}^r \dot{x}^s = (R_{rijs} - R_{jrís}) \dot{x}^r \dot{x}^s$$

which, by increasing indices, implies $(R^i_{\,jr} - R^i_{\,jr}) \dot{x}^r \dot{x}^s = 0$. \hfill \Box

Lemma 3.9. Let $X_i, i = 1, \ldots, 5$, be vector fields on $M$. Then the conditions (A) and (B) imply

$$\nabla(g \otimes \bar{g}) = 0.$$  

and

$$\sum g(X_1, X_4)g(R(X_2, X_3)(X_4), X_4) = 0,$$
which proves the first part of Lemma 3.10.

**Proof.** For any vector fields $X_1, \ldots, X_5$ we have

$$\nabla_{X_5}(g \otimes \tilde{g})(X_1, X_2, X_3, X_4) = \tilde{g}(X_1, X_2, X_3, X_4) \nabla_{X_5}g(X_1, X_2, X_3, X_4) + g(X_1, X_2, X_3, X_4) \nabla_{X_5}\tilde{g}(X_1, X_2, X_3, X_4)$$

which follows from

$$\nabla_{X_5}\tilde{g}(X_1, X_2, X_3, X_4) = -\nabla_{X_5}g(X_1, X_2, X_3, X_4).$$

(3.8) now follows from the fact that $g \otimes \nabla g$ is a symmetric tensor, i.e. $g \otimes \nabla g : M \to \otimes^5 T^* M$.

To prove (3.9) we consider the covariant derivative of the equation

$$g(X_1, X_4)(\nabla_{X_2}g)(X_4, X_4) - g(X_2, X_4)(\nabla_{X_1}g)(X_4, X_4) = 0$$

with respect to $X_3$. Then taking the sum with respect to the cyclic permutation of $X_1, X_2, X_3$, (A), (B) and

$$(\nabla^2 g)(X_4, X_4, X_1, X_2) - (\nabla^2 g)(X_4, X_4, X_2, X_1) = -2g(R(X_1, X_2)(X_4), X_4)$$

we get (3.9). \hfill \Box

**Lemma 3.10.** The conditions (A) and (B) imply

(3.10) $$(\nabla g)(u, u)(u)u^V \wedge \Lambda(g, K) = 0$$

and

(3.11) $$[\Lambda(g, K), u^H] \wedge u^V = 0.$$  

**Proof.** In coordinates (3.8) reads as

$$g^{pq} \nabla_m g_{rs} = -g_{rs} \nabla_m g^{pq}.$$  

Then we have

$$(\nabla g)(u, u)(u)u^V \wedge \Lambda(g, K)$$

$$= (\nabla_m g_{rs})\hat{x}^r \hat{x}^s \hat{x}^m \hat{x}^j g^{jk} (\delta^i_q \partial_i \wedge \partial_j \wedge \partial_k + K_p^{iq} \hat{x}^p \partial_i \wedge \partial_j \wedge \partial_k)$$

which vanishes if and only if

$$0 = \hat{x}^r \hat{x}^s \hat{x}^m (\hat{x}^j g^{jk} - \hat{x}^k g^{ij}) \nabla_m g_{rs} = \hat{x}^r \hat{x}^s \hat{x}^m (\hat{x}^k g_{rs} \nabla_m g^{ij} - \hat{x}^j g_{rs} \nabla_m g^{ik})$$

$$= 2h(u) \hat{x}^m (\hat{x}^k \nabla_m g^{ij} - \hat{x}^j \nabla_m g^{ik}).$$

But

(3.12) $$\hat{x}^m (\hat{x}^k \nabla_m g^{ij} - \hat{x}^j \nabla_m g^{ik}) = \hat{x}^m \hat{x}^s (g^{jp} g^{kq} \nabla_m g_{ps} - g^{kp} g^{jq} \nabla_m g_{ps})$$

$$= \hat{x}^m \hat{x}^s g^{ip} g^{qu} \nabla_{ip} g^{kq} (g_{ut} \nabla_m g_{ps} - g_{pu} \nabla_m g_{ts})$$

$$= \hat{x}^m \hat{x}^s g^{ip} g^{qu} \nabla_{ip} g^{kq} (g_{ut} \nabla_p g_{rs} - g_{pu} \nabla_t g_{rs})$$

which proves the first part of Lemma 3.10.

In coordinates

$$[\Lambda(g, K), u^H] \wedge u^V = \hat{x}^r \hat{x}^k \nabla_r g^{mj} (\delta^i_m \partial_i \wedge \partial_j \wedge \partial_k + K^i_s \hat{x}^s \partial_i \wedge \partial_j \wedge \partial_k)$$

$$+ R^i_{rj} \hat{x}^r \hat{x}^k \hat{x}^i \partial_j \wedge \partial_j \wedge \partial_k.$$
The first term on the right hand side vanishes because of (3.12). The second term vanishes because of (3.9). Really in coordinates (3.9) reads as
\[ \dot{x}^r \dot{x}^s (g_{ir} R_{sjkt} + g_{jr} R_{skit} + g_{kr} R_{sijt}) = 0 \]
which, by using (3.6) and increasing indices, implies
\[ \dot{x}^r \dot{x}^s (\dot{x}^i R^j_{rs} - \dot{x}^i R^k_{rs} - \dot{x}^j R^k_{rs} + \dot{x}^k R^j_{rs}) = 0. \]

4. Classification of natural 2-vector fields generated by $g$ and $K$

Any 2-vector field on $TM$ has coordinate expression $\Lambda = \Lambda^i \partial_i + \Lambda^j \partial_j$. The canonical 2-vector field $\Lambda(g, K)$ is given by $\Lambda^i = 0$, $\Lambda^j = g^j$, and $\Lambda^k = (g^m K^i_m - g^m K^j_m) \dot{x}^r$. $\Lambda(g, K)$ is a natural 0-order operator from $TM \times_M reg \otimes^2 T^*M \times_M C_\tau M$ into $\wedge^2 T(TM)$ projectable over the identity of $TM$. In this section we shall classify all operators of this type.

Let us recall the standard fibre $Q$ of the functor $T \times reg \otimes^2 T^* \times C_\tau$ with the action of the group $G_n^2$ on $Q$ given by (2.4) and the fundamental vector fields (2.5) and (2.6).

Let us denote by $S = \mathbb{R}^n \times \wedge^2 \mathbb{R}^{2n}$ the standard fibre of $\wedge^2 T(T)$ with the canonical coordinates $(\dot{x}^i, (\Lambda_1, \Lambda_2, \Lambda_3))$, $\Lambda_1 = -\Lambda_1$, $\Lambda_3 = -\Lambda_3$. The action of $G_n^2$ on $S$ is given by
\[
(4.1) \quad \dot{x}^i = a^i p^j, \quad \ddot{\Lambda}^i_1 = a^i_1 a^j_1 \Lambda_1^{pq}, \quad \ddot{\Lambda}^i_2 = a^i_2 a^j_2 \Lambda_2^{pq} + a^i_2 a^j_3 \dot{x}^r \Lambda_1^{pq},
\]
\[
\ddot{\Lambda}^i_3 = a^i_3 a^j_3 \Lambda_3^{pq} + (a^i_3 a^j_3 - a^i_2 a^j_2) \dot{x}^r \Lambda_2^{pq} + a^i_2 a^j_3 \dot{x}^r \dot{x}^s \Lambda_1^{pq}.
\]

The fundamental vector fields on $S$ relative to this action are
\[
(4.2) \quad \xi^q_1(S) = \dot{x}^q \frac{\partial}{\partial \dot{x}^p} + 2\Lambda_1^{pq} \frac{\partial}{\partial \Lambda_1^{pq}} + \Lambda_2^{pq} \frac{\partial}{\partial \Lambda_2^{pq}} + 2\Lambda_3^{pq} \frac{\partial}{\partial \Lambda_3^{pq}},
\]
\[
(4.3) \quad 2\xi^q_p(S) = \Lambda_1^{qr} \dot{x}^q \frac{\partial}{\partial \Lambda_2^{rp}} + \Lambda_1^{iq} \dot{x}^r \frac{\partial}{\partial \Lambda_2^{ip}} + 2(\Lambda_2^{qj} \dot{x}^r + \Lambda_2^{ij} \dot{x}^q) \frac{\partial}{\partial \Lambda_3^{pj}}.
\]

First we shall prove

**Theorem 4.1.** All $G_n^2$-equivariant mappings $F : Q \to S$ (over the identity of $\mathbb{R}^n$) are given by
\[
F_1^{ij} = \gamma_1^{ij},
\]
\[
F_2^{ij} = \gamma_1^{ir} K_{r}^{j} q^{i} + \gamma_2^{ij},
\]
\[
F_3^{ij} = \gamma_1^{iq} K_{p}^{i} q^{j} m^{i} \dot{x}^{k} \dot{x}^{m} + (\gamma_2^{qj} K_{q}^{j} k - \gamma_2^{qj} K_{q}^{i} k) \dot{x}^{k} + \gamma_3^{ij},
\]
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where $\gamma_{ij}^{\alpha}$ are functions on $Q$ which are solutions of the following system of partial differential equations

\begin{align}
\frac{\partial \tilde{\zeta}_{ij}}{\partial K_{pq}^r} &= 0, \\
\dot{x}^q \frac{\partial \tilde{\zeta}_{ij}}{\partial \dot{x}^p} - 2g_{rp} \frac{\partial \tilde{\zeta}_{ij}}{\partial g_{rq}} &= \tilde{\zeta}_{iq} \delta_p^j + \tilde{\zeta}_{ij} \delta_p^i,
\end{align}

Moreover, $\gamma_{1}^{ij} = -\gamma_{j}^{ji}, \gamma_{3}^{ij} = -\gamma_{3}^{ji}$.

**Proof.** A mapping $F: Q \to S$ is $G_2^n$-equivariant if and only if the corresponding fundamental vector fields (2.5), (2.6) and (4.2), (4.3) are $F$-related. If $F$ has the coordinate expression

$$\dot{x}^i = \dot{x}^i, \quad \Lambda_{ij}^{\alpha} = F_{ij}^{\alpha}(\dot{x}^p, g_{pq}, K_{qr}), \quad \alpha = 1, 2, 3,$$

then $F_{ij}^{\alpha}$ have to satisfy the following system of partial differential equations

\begin{align}
\dot{x}^q \frac{\partial F_{ij}^{\alpha}}{\partial \dot{x}^p} - 2g_{rp} \frac{\partial F_{ij}^{\alpha}}{\partial g_{rq}} + \left(\delta_p^q K_{st} - \delta_r^q K_{sp} - \delta_t^q K_{st} \right) \frac{\partial F_{ij}^{\alpha}}{\partial K_{pq}^r} &= F_{ij}^{\alpha} \delta_p^j + F_{ij}^{\alpha} \delta_p^i, \\
\frac{\partial F_{ij}^{1}}{\partial K_{q}^p} &= 0, \\
\frac{\partial F_{ij}^{2}}{\partial K_{q}^p} &= \frac{1}{2} \left( F_{ij}^{1r} \dot{x}^q \delta_p^j + F_{ij}^{1q} \dot{x}^r \delta_p^i \right), \\
\frac{\partial F_{ij}^{3}}{\partial K_{q}^p} &= \frac{1}{2} \left( F_{ij}^{2q} \dot{x}^r \delta_p^i + F_{ij}^{2r} \dot{x}^q \delta_p^j - F_{ij}^{2q} \dot{x}^r \delta_p^j - F_{ij}^{2r} \dot{x}^q \delta_p^j \right),
\end{align}

Now we have to prove that all solutions of (4.7) - (4.10) are of the form (4.4) where (4.5) and (4.6) are satisfied.

Substituting (4.8) into (4.7), where $\alpha = 1$, we get that

$$F_{ij}^{1} = \gamma_{ij}^{1}, \quad \gamma_{ij}^{1} = -\gamma_{ji}^{1},$$

and $\gamma_{ij}^{1}$ satisfies (4.5) and (4.6).

Let us rewrite (4.9) in the form

$$\frac{\partial F_{ij}^{2}}{\partial K_{q}^p} = \frac{1}{2} F_{ij}^{ab} \dot{x}^a \delta_b^j \left( \delta_d^r \delta_a^p + \delta_d^r \delta_a^q \right),$$

i.e. in the form

$$\frac{\partial F_{ij}^{2}}{\partial K_{q}^p} = F_{ij}^{ab} \dot{x}^a \partial K_{q}^b \partial K_{q}^p.$$

Integrating this formula and substituting (4.11) we get

$$F_{ij}^{2} = \gamma_{ij}^{1} K_{r}^q \dot{x}^r + \gamma_{ij}^{2}.$$
Finally (4.10) can be rewritten in the form
\[
\frac{\partial F_{ij}^{ij}}{\partial K_{q}^{pr}} = \frac{1}{2} F_{1}^{aq} \dot{x}^{b} \delta_{p}^{q} (\delta_{b}^{q} \delta_{a}^{r} + \delta_{b}^{r} \delta_{a}^{q}) - \frac{1}{2} F_{2}^{aq} \dot{x}^{b} \delta_{p}^{q} (\delta_{b}^{q} \delta_{a}^{r} + \delta_{b}^{r} \delta_{a}^{q}),
\]

i.e. in the form
\[
\frac{\partial F_{ij}^{ij}}{\partial K_{q}^{pr}} = F_{2}^{aq} \dot{x}^{b} \delta_{p}^{q} \delta_{a}^{b} \delta_{b}^{r} - F_{2}^{aq} \dot{x}^{b} \delta_{p}^{q} \delta_{b}^{r} \delta_{a}^{b}.
\]

Integrating this formula and substituting (4.12) we get
\[
F_{3}^{ij} = \dot{\gamma}_{3}^{ij} K_{i}^{p} K_{j}^{q} \dot{x}^{k} \dot{x}^{m} + (\dot{\gamma}_{2}^{ij} K_{i}^{j} - \dot{\gamma}_{2}^{ij} K_{j}^{i}) \dot{x}^{k} + \dot{\gamma}_{3}^{ij},
\]
where \(\dot{\gamma}_{3}^{ij} = -\dot{\gamma}_{3}^{ij}\) and \(\dot{\gamma}_{3}^{ij}\) satisfies (4.5) and (4.6).

According to Theorem 4.1 to classify all natural operators from \(TM \times M \reg \otimes T^*M \times M \) it is sufficient to classify all operators with equivariant mappings expressed by (4.5) and (4.6), i.e. operators from \(TM \times M \reg \otimes T^*M \) to \(TM \otimes TM\). To classify all such operators we shall use the following Theorem, [5, 4],

**Theorem 4.2.** Let \((M, g)\) be an oriented pseudo-Riemannian manifold of dimension \(n > 3\). Then all natural operators from \(TM \times M \reg \otimes T^*M \) to \(T^*M \otimes T^*M\) (natural F-metrics) are symmetric and are of the form
\[
\beta_{u}(X, Y) = \mu(h(u)) g(X, Y) + \nu(h(u)) g(X, u) g(Y, u)
\]
where \(X, Y\) are vector fields and \(\mu, \nu\) are smooth real functions defined on \(\mathbb{R}\).

In coordinates
\[
\beta_{ij} = \mu(h(u)) g_{ij} + \nu(h(u)) g_{ip} g_{jq} \dot{x}^{p} \dot{x}^{q}.
\]

Now we can prove the inverse version of the above Theorem 4.2.

**Theorem 4.3.** Let \((M, g)\) be an oriented pseudo-Riemannian manifold of dimension \(n > 3\). Then all natural operators from \(TM \times M \reg \otimes T^*M \) to \(TM \otimes TM\) are symmetric and are of the form
\[
\gamma_{u}(\rho, \sigma) = \gamma_{1}(h(u)) \tilde{g}(\rho, \sigma) + \gamma_{2}(h(u)) \rho(u) \sigma(u)
\]
where \(\rho, \sigma\) are 1-form fields and \(\gamma_{1}, \gamma_{2}\) are smooth real functions defined on \(\mathbb{R}\).

In coordinates
\[
\gamma^{ij} = \gamma_{1}(h(u)) g^{ij} + \gamma_{2}(h(u)) \dot{x}^{i} \dot{x}^{j}.
\]

**Proof.** Any natural operator \(TM \times M \reg \otimes T^*M \) to \(TM \otimes TM\) can be interpreted as a natural real function on \(TM \times M \reg \otimes T^*M \times M \) bilinear on \(T^*M\). Similarly any natural F-metric of Theorem 4.2 is a natural real function on \(TM \times M \times M \reg \otimes T^*M\) bilinear on \(TM\). Theorem 4.3 now follows from the classification of Theorem 4.2 and the fact, [2], that all natural isomorphisms \(T^*M \rightarrow TM\) induced by the metric \(g\) are of the form
\[
X = \kappa(h(u)) \tilde{g}^{\rho}(\rho), \quad X^{i} = \kappa(h(u)) g^{ij} \rho_{j},
\]
where $\rho$ is a 1-form field and $\kappa$ is a smooth real function defined on $\mathbb{R}$ such that $\kappa(t) \neq 0$ for any $t \in \mathbb{R}$. The operator $\gamma$ is now a composition of the operator (4.17) and natural $F$-metric $\beta$, i.e. in coordinates

\begin{equation}
\gamma^{ij} = \kappa^2(h(u))g^{ip}g^{jq}(\mu(h(u))g_{pq} + \nu(h(u))g_{pr}g_{qs}\dot{x}^r\dot{x}^s)
= \gamma_1(h(u))g^{ij} + \gamma_2(h(u))\dot{x}^i\dot{x}^j,
\end{equation}

where $\gamma_1 = \kappa^2\mu, \gamma_2 = \kappa^2\nu$. (4.18) is just the equivariant mapping corresponding to (4.15).

**Remark 4.4.** In Theorems 4.2 and 4.3 we have restricted the dimension of the underlying manifold on $n > 3$ because our standard model is an Lorentzian manifold of dimension 4 (spacetime). But both Theorems 4.2 and 4.3 are correct for non-oriented manifolds if $n \geq 2$. For oriented manifolds in dimensions 2 and 3 there are also antisymmetric natural $F$-metrics and so antisymmetric natural operators $\gamma$ (see [5]).

Now we can classify all natural 2-vector fields. We have

**Theorem 4.5.** Let $(M, g)$ (dim $M > 3$) be an oriented pseudo-Riemannian manifold endowed with a symmetric linear connection $K$. Then all natural operators from $TM \times M \otimes T^*M \times M C\tau M$ into $\wedge^2 T(TM)$ projectable over the identity of $TM$ are of the form

\begin{equation}
\Lambda(\gamma, K) = K\bar{\wedge}\gamma \partial,
\end{equation}

where $\bar{\wedge}\gamma$ denotes the wedge product followed by the contraction through the operator $\gamma$ of Theorem 4.3. In coordinates

\begin{equation}
\Lambda(\gamma, K) = (\gamma_1(h(u))g^{ij} + \gamma_2(h(u))\dot{x}^i\dot{x}^j)(\partial_i + K^{m}_{ik}\dot{x}^k\partial_m) \wedge \partial_j,
\end{equation}

where $\gamma_1, \gamma_2$ are smooth real functions defined on $\mathbb{R}$.

**Proof.** By Theorem 4.3 we have $\gamma_1^{ij} = \gamma_3^{ij} = 0$ in Theorem 4.1. So we have the equivariant mappings $F : Q \to S$ corresponding to our operators in the form

\begin{align}
F_1^{ij} &= 0, & F_2^{ij} &= \gamma_2^{ij}, & F_3^{ij} &= (\gamma_2^{ij}K^{i}_{q}^{k} - \gamma_2^{qi}K^{i}_{q}^{k})\dot{x}^k,
\end{align}

where

\begin{equation}
\gamma_2^{ij} = \gamma_1(h(u))g^{ij} + \gamma_2(h(u))\dot{x}^i\dot{x}^j.
\end{equation}

It is easy to see that it is just the mapping corresponding to (4.19).

**Remark 4.6.** The canonical 2-vector field from Section 3 corresponds to $\gamma_1 \equiv 1, \gamma_2 \equiv 0$. From (2.1) and (2.2) it follows that the 2-vector field corresponding to $\gamma_1 \equiv 0, \gamma_2 \equiv 1$ is $u^H \wedge u^V$. Then (4.19) can be written in the form

\begin{equation}
\Lambda(\gamma, K) = \gamma_1(h(u))\Lambda(g, K) + \gamma_2(h(u)) u^H \wedge u^V
\end{equation}

which is much more convenient for our further purposes.
Remark 4.7. The 2-vector field $\Lambda(\gamma, K)$ can be characterized by
\[
\Lambda(\gamma, K)(\rho^V, \sigma^V) = 0, \quad \Lambda(\gamma, K)(\rho^V, \sigma^H) = \gamma(\rho, \sigma),
\]
\[
\Lambda(\gamma, K)(\rho^H, \sigma^V) = -\gamma(\rho, \sigma), \quad \Lambda(\gamma, K)(\rho^H, \sigma^H) = 0,
\]
where $\rho, \sigma$ are 1-form fields on $M$, $\rho^V, \sigma^V$ are their vertical lifts (pullbacks) and $\rho^H, \sigma^H$ are horizontal lifts with respect to $K$.

5. Natural Poisson structures

In this Section we shall recall, [3], conditions for $\Lambda(\gamma, K)$ to define a Poisson structure on $TM$, i.e. we have to find conditions for $\Lambda(\gamma, K)$ to satisfy (1.2).

Lemma 5.1. The 2-vector field $\Lambda(\gamma, K)$ is of constant maximal rank if and only if $\gamma^1(t) \neq 0$ and $\gamma^1(t) + 2t\gamma^2(t) \neq 0$ for any $t \in \mathbb{R}$.

Proof. $\Lambda(\gamma, K)$ is of constant maximal rank if and only if the matrix $\gamma^1(h) g^{ij} + \gamma^2(h) \dot{x}^i \dot{x}^j$ is regular. It is easy to see that it is so if and only if $\gamma^1(t)$ and $\gamma^1(t) + 2t\gamma^2(t)$ are everywhere nonvanishing functions.

Lemma 5.2. We have
\[
[\Lambda(\gamma, K), \Lambda(\gamma, K)] = \gamma^2_2(h) [\Lambda(g, K), \Lambda(g, K)]
\]
\[
+ \gamma^1(h) \gamma^1_2(h) ((\nabla g)(u, u)^V) \wedge \Lambda(g, K)
\]
\[
+ \gamma^1(h) \gamma^2_2(h) ((\nabla g)(u, u)^V) \wedge u^H \wedge u^V
\]
\[
+ 2(\gamma^1(h) \gamma^2_2(h) - \gamma^1(h) \gamma^1_2(h) - 2h \gamma^2(h) \gamma^1_2(h)) u^H \wedge \Lambda(g, K)
\]
\[
+ \gamma^2_2(h) \gamma^1(h) (\nabla g)(u, u)(u) u^V \wedge \Lambda(g, K)
\]
\[
+ 2\gamma^1(h) \gamma^2_2(h) [\Lambda(g, K), u^H] \wedge u^V.
\]

Proof. It is easy to prove it by using (4.22) and properties of the Schouten-Nijenhuis bracket.

Theorem 5.3. The nondegenerate 2-vector field $\Lambda(\gamma, K)$ defines a Poisson structure on $TM$ if and only if the conditions (A), (B) and

(C) $\gamma^1(t) \gamma^2(t) - \gamma^1(t) \gamma^1_2(t) - 2t \gamma^2(t) \gamma^1_2(t) = 0$

are satisfied for any $t \in \mathbb{R}$.

Proof. $\Leftarrow$ It follows from Theorem 3.4, Lemma 3.7 and Lemma 3.10.

$\Rightarrow$ All 2-vector fields on the right hand side of Lemma 5.2 are independent, i.e. $[\Lambda(\gamma, K), \Lambda(\gamma, K)] = 0$ if and only if all terms on the right hand side vanish. Since $\gamma^1(h) \neq 0$ the first term vanishes (by Theorem 3.4) if and only if the condition (A) is satisfied. The second and the third terms vanish if and only if the condition (B) is satisfied (Lemma 3.7) and the fourth term vanishes if and only if the condition (C) is satisfied. The others terms vanish because of the conditions (A) and (B) (Lemma 3.10).
Now, let us compare the conditions of Theorem 5.3 for natural Poisson structures with conditions for natural symplectic structures given in [1]. First we recall that any 2-form field $\Omega(\beta, K)$ on $TM$, dim$M > 3$, naturally given by $g$ and $K$ can be characterized by

$$\Omega(\beta, K)(X^H, Y^H) = 0, \quad \Omega(\beta, K)(X^H, Y^V) = -\beta(X, Y), \quad \Omega(\beta, K)(X^V, Y^V) = 0. \tag{5.1}$$

where $X, Y$ are vector fields on $M$ and $\beta$ is a natural $F$-metric given by Theorem 4.2. In coordinates

$$\Omega(\beta, K) = (\mu(h(u)) g_{mj} + \nu(h(u)) g_{mr} g_{js} \dot{x}^r \dot{x}^s)(\dot{d}^m - K^m_k \dot{x}^k d^i) \wedge d^j.$$

Then we have, [1],

**Theorem 5.4.** $\Omega(\beta, K)$ is a symplectic form on $TM$ if and only if $\nu = \dot{\mu}$ and the real smooth function $\mu$ satisfies

$$\mu(t) \neq 0, \quad \mu(t) + 2t \dot{\mu}(t) \neq 0 \tag{5.2}$$

for all $t \in \mathbb{R}$. Moreover $g$ and $K$ have to satisfy conditions (A) and (B). \hfill \Box

Let us suppose that the 2-vector field $\Lambda(\gamma, K)$ is the Poisson 2-vector field given by the symplectic 2-form field $\Omega(\beta, K)$. It is easy to see that in this case $\gamma$ (given by (4.16)) is the inverse of the natural $F$-metric $\beta$ (given by Theorem 4.2), i.e. $\beta_{ik} \gamma^{kj} = \delta^j_i$. Then

$$\mu(t) \gamma_1(t) = 1, \quad \mu(t) \gamma_2(t) + \nu(t) \gamma_1(t) + 2t \nu(t) \gamma_2(t) = 0. \tag{5.3}$$

If the conditions of Theorem 5.4 are satisfied, then $\mu = \frac{1}{\gamma_1}$ and $\nu = -\frac{\dot{\gamma}_1}{\gamma_1}$ (from the condition $\nu = \dot{\mu}$ for symplectic forms). Substituting these equalities into second term of (5.3) we get the condition (C) of Theorem 5.3. On the other hand if $\Omega(\beta, K)$ is a symplectic 2-form field given by the nondegenerate Poisson 2-vector field $\Lambda(\gamma, K)$, then the conditions of Theorem 5.3 induce in the same way conditions of Theorem 5.4. So the conditions for natural symplectic and natural Poisson structures on $TM$ are equivalent.

**Remark 5.5.** In Theorem 5.3 we have supposed $\Lambda(\gamma, K)$ to be nondegenerate. If we admit also 2-vector fields $\Lambda(\gamma, K)$ which are not of maximal rank we get from Lemma 5.2 that all such 2-vector fields are characterized by $\gamma_1 \equiv 0$, i.e. they are of the form $\Lambda(\gamma, K) = \gamma_2(h(u)) u^H \wedge u^V$, where $\gamma_2$ is arbitrary smooth real function on $\mathbb{R}$.

6. **Natural Jacobi structures**

In this Section we shall recall, [3], conditions for $E, \Lambda(\gamma, K)$ to define a Jacobi structure on $TM$, i.e. we have to find conditions for $E, \Lambda(\gamma, K)$ to satisfy (1.3).
Lemma 6.1. We have
\[ [E, \Lambda(\gamma, K)] = \alpha(h)\gamma_1(h) [u^H, \Lambda(g, K)] + \left( \frac{1}{2} \alpha(h)\gamma_1(h) \nabla g(u, u)(u) + 2h\beta(h)\gamma_1(h) - \beta(h)\gamma_1(h) \right) \Lambda(g, K) \]
\[ + \left( \frac{1}{2} \alpha(h)\gamma_2(h) \nabla g(u, u)(u) + 2h\beta(h)\gamma_2(h) + \beta(h)\gamma_2(h) \right) \]
\[ - 2h\dot{\beta}(h)\gamma_2(h) - \frac{1}{2} \dot{\alpha}(h)\gamma_2(h) \nabla g(u, u)(u) - \dot{\beta}(h)\gamma_1(h) \right) u^H \wedge u^V \]
\[ + \frac{1}{2} \dot{\alpha}(h)\gamma_1(h)(\nabla g(u, u)^V) \wedge u^H + \frac{1}{2} \dot{\beta}(h)\gamma_1(h)(\nabla g(u, u)^V) \wedge u^V . \]

Proof. It is easy to prove Lemma 6.1 by using properties of the Schouten-Nijenhuis bracket.

Lemma 6.2. Let \( \Lambda(\gamma, K) \) be of maximal rank. We have
\[ [E, \Lambda(\gamma, K)] = 0 \]
if and only if one of the following groups of conditions is satisfied:
I. \( \alpha \equiv 0, \beta \equiv 0; \)
II. \( \nabla g = 0 \) and the equalities
\[ \beta(t)(\gamma_1(t) - 2t\dot{\gamma}_1(t)) = 0 , \]
\[ \beta(t)(\gamma_2(t) + 2t\dot{\gamma}_2(t)) - \dot{\beta}(t)(\gamma_1(t) + 2t\dot{\gamma}_2(t)) = 0 \]
are satisfied for all \( t \in \mathbb{R} \).

Proof. All terms on the right hand side in Lemma 6.1 have to vanish. The first term vanishes if and only if \( \alpha \equiv 0 \) or \( [u^H, \Lambda(g, K)] = 0 \).

First let us suppose \( [u^H, \Lambda(g, K)] = 0 \) which is equivalent by Lemma 3.8 with \( \nabla g = 0 \), i.e. \( K \) is the metric connection. The last two terms in Lemma 6.1 vanish and the second and third terms vanish if and only if (6.1) and (6.2) are satisfied and we have II.

Now, let us suppose \( \alpha \equiv 0 \) and \( \nabla g \neq 0 \). Then the second and the third terms vanish if and only if (6.1) and (6.2) are satisfied and the last term vanishes if and only if \( \dot{\beta} \equiv 0 \), i.e. \( \beta \) is a constant function. Then for \( \beta \neq 0 \) (6.1) is equivalent with
\[ \gamma_1(t) = 2t\dot{\gamma}_1(t), \quad \text{i.e.} \quad \gamma_1(t) = \sqrt{t} \]
\[ \gamma_2(t) = -2t\dot{\gamma}_2(t), \quad \text{i.e.} \quad \gamma_2(t) = \frac{1}{\sqrt{t}} \]
and similarly (6.1) is equivalent with
which is in the contradiction with the globality of \( \gamma_1, \gamma_2 \). Hence (6.1) and (6.2) are satisfied only for \( \beta \equiv 0 \) and we have I.

Lemma 6.3. Let \( \Lambda(\gamma, K) \) be of maximal rank. We have
\[ [\Lambda(\gamma, K), \Lambda(\gamma, K)] = 2E \wedge \Lambda(\gamma, K) \]
if and only if the conditions (A), (B) and
\begin{equation}
\alpha(t)\gamma_1(t) = \gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_1(t) - 2t\gamma_2(t)\dot{\gamma}_1(t), \quad \beta(t) = 0
\end{equation}
are satisfied for any \( t \in \mathbb{R} \).

**Proof.** We have
\[ 2E \wedge \Lambda(\gamma, K) = 2\alpha\gamma_1 u^H \wedge \Lambda(g, K) + 2\beta\gamma_1 u^V \wedge \Lambda(g, K). \]
Comparing this with expression of \([\Lambda(\gamma, K), \Lambda(\gamma, K)]\) from Lemma 5.2 we get
\begin{align}
\[ & [\Lambda(g, K), \Lambda(g, K)] = 0, \\
\[ & (\nabla g)(u, u)\gamma_1(u) - \alpha(h(u))\dot{\gamma}_2(h(u)) - \beta(h(u))\dot{\gamma}_2(h(u)) = 0, \\
\[ & \gamma_2[\Lambda(g, K), u^H] \wedge u^V = 0. \]
\end{align}
By Theorem 3.4 (6.6) holds if and only if (A) is satisfied, (6.7) is equivalent with (B). (A) and (B) imply (6.8) and (6.11) and (6.10) is reduced by Lemma 3.10 to \( \beta \equiv 0 \).

**Theorem 6.4.** Let \( \Lambda(\gamma, K) \) be of maximal rank and \( E \) be a non-zero vector field. \( E \) and \( \Lambda(\gamma, K) \) define a Jacobi structure on \( TM \) if and only if \( K \) is the metric connection and \( E = \alpha(h) u^H \), where
\begin{equation}
\alpha(t)\gamma_1(t) = \gamma_1(t)\gamma_2(t) - \gamma_1(t)\dot{\gamma}_1(t) - 2t\gamma_2(t)\dot{\gamma}_1(t)
\end{equation}
for any \( t \in \mathbb{R} \).

**Proof.** Theorem 6.4 follows from Lemma 6.1 and Lemma 6.3.

**Remark 6.5.** If \( E \) is the zero vector field then Theorem 6.4 reduces to Theorem 5.3 and the Jacobi structures reduces to the Poisson structure.

**Remark 6.6.** In Theorem 6.4 we have supposed \( \Lambda(\gamma, K) \) to be of maximal rank. If we admit also 2-vector fields which are not of maximal rank we get from Lemma 6.3 that all such 2-vectors are characterized by \( \gamma_1 \equiv 0 \). Lemma 6.1 then implies that the functions \( \alpha, \beta \) and \( \gamma_2 \) have to satisfy
\begin{align*}
\frac{1}{2} \nabla g(u, u)(u)(\alpha(h(u))\dot{\gamma}_2(h(u)) - \dot{\alpha}(h(u))\gamma_2(h(u))) \\
+ 2h(u)(\beta(h(u))\dot{\gamma}_2(h(u)) - \dot{\beta}(h(u))\gamma_2(h(u))) + \beta(h(u))\gamma_2(h(u)) = 0.
\end{align*}
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