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Simplicial types and polynomial algebras


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SIMPLICIAL TYPES AND POLYNOMIAL ALGEBRAS

FRANCISCO GÓMEZ

Abstract. This paper shows that the simplicial type of a finite simplicial complex \( K \) is determined by its algebra \( A \) of polynomial functions on the baricentric coordinates with coefficients in any integral domain. The link between \( K \) and \( A \) is done through certain admissible matrix associated to \( K \) in a natural way. This result was obtained for the real numbers by I. V. Savel'ev [5], using methods of real algebraic geometry. D. Kan and E. Miller had shown in [2] that \( A \) determines the homotopy type of the polyhedron associated to \( K \) and not only its rational homotopy type as it was previously proved by D. Sullivan in [6].

§1. Introduction

D. Kan and E. Miller [2] proved that for every finite simplicial complex \( K \) and any unique factorization domain with unit \( R \) the Sullivan’s algebra of polynomial 0-forms with coefficients in \( R \), \( A_0^0(R)(K) \), determines the homotopy type of the associated polyhedron \( |K| \) and not only its rational homotopy type as was previously proven by D. Sullivan [6]. Later I. V. Savel’ev, using methods of real algebraic geometry, proved in a paper published in 1991, [5], that actually one can deduce from \( A_0^0(R)(K) \), for \( R \) being the real numbers, the whole structure of the simplicial complex \( K \) and not just its homotopy type.

The purpose of this paper is to show by a different method that the use of the real field is not essential and any integral domain \( R \) could be used to recover from \( A_0^0(R)(K) \) the simplicial complex \( K \), up to simplicial equivalence, and therefore contains all the information about the topological type of \( |K| \), see Theorem (3.9).

The link between the finite simplicial complex \( K \) and its algebra of polynomial 0-forms \( A_0^0(R)(K) \) is done here through a certain admissible matrix \( \varphi_K \), associated to \( K \) in a natural way.

Corollary (3.7) of this paper gives also a more direct proof of how to obtain Sullivan’s de Rham complex from its 0-forms, see [3], and the Example (3.8) shows

2000 Mathematics Subject Classification: 58A10, 55U10, 14F40.

Key words and phrases: simplicial complex, algebraic de Rham complex, Sullivan’s de Rham complex.

Partially supported by a DGICYT grant (PB97-1095) and a Junta de Andalucía grant (FQM0213).

Received July 10, 2000.
that the cohomology of the algebraic de Rham complex of $A^n_R(K)$ does not give the “correct” cohomology of $|K|$.

In this paper $R$ will be any commutative integral domain with unit and algebras are supposed to be commutative with a unit which is preserved by morphisms.

Exterior power is denoted by $\Lambda$ and if $A$ is an $R$–algebra, $\Omega_R(A)$ denotes then the $A$–module of Kähler differentials, i.e. ker $\mu/(\ker \mu)^2$ where $\mu: A \otimes_R A \to A$ is the multiplication.

We have $d: A \to \Omega_R(A)$ given by $da =$class of $(a \otimes 1 - 1 \otimes a)$ and the standard extension of $d$ to obtain the algebraic de Rham complex on $A$, $(\Lambda^*_A \Omega_R(A), d)$.

§2. Simplicial complexes and admissible matrices

Let $K$ be a finite simplicial complex with maximal simplices $\sigma_1, \ldots, \sigma_r$ and let us denote by $\mathcal{P}_K$ the set of simplices appearing as intersections of maximal simplices of $K$ and so the simplices of $K$ are subsets of members of $\mathcal{P}_K$.

Define a partition $\Sigma_K$ of $K$ by specifying that two vertices $v$ and $w$ are in the same class if and only if for each maximal simplex $\sigma_i$ either $\{v, w\} \subset \sigma_i$ or $\{v, w\} \cap \sigma_i = \emptyset$.

It is clear that $\Sigma_K$ can be regarded as the set of vertices of a simplicial complex with maximal simplices $\tilde{\sigma}_i = \{\omega \in \Sigma_K | \omega \subset \sigma_i\}, i = 1, \ldots, r$. Then $\Sigma \Sigma_K = \Sigma_K$ and we have simplicial maps $f: K \to \Sigma_K$ and $g: \Sigma_K \to K$ such that $f \circ g$ is the identity and $g \circ f$ induces a map homotopic to the identity in the associated polyhedron. In fact, just define $f(v)$ as the member of $\Sigma_K$ containing $v$ and $g(w) \in w$ for all $w \in \Sigma_K$.

(2.1) Observe that if $\sigma \in \mathcal{P}_K$ and we consider $\tilde{\sigma} = \sigma - \cup_{\sigma_i \neq \sigma} \sigma_i$, either $\tilde{\sigma} = \emptyset$ or $\tilde{\sigma} \in \Sigma_K$ and clearly $\Sigma_K$ coincides with the set of nonempty $\tilde{\sigma}$ for all $\sigma \in \mathcal{P}_K$.

The following formula is deduced easily for the number of elements of $\tilde{\sigma}$

$$|\tilde{\sigma}| = \sum_\omega (-1)^{|\omega|} |\omega|$$

where $\omega$ in the sum runs through the members of $\mathcal{P}_K$ of the form $\sigma \cap \sigma_i$, for $i = 1, \ldots, r$ and $| \ |$ denotes number of elements.

(2.2) Define then a matrix $\varphi_K = (a_{ij})$ of $r$ rows and $s$ columns by specifying that $a_{ij}$ is either 1 or 0 according to $\omega_j$ being or not a subset of $\sigma_i$.

Here $\Sigma_K = \{\omega_1, \ldots, \omega_s\}$.

We also consider the integer vector $\mathbf{n}_K = (n_1, \ldots, n_s)$ where $n_i$ is the number of elements of $\omega_i, i = 1, \ldots, s$.

We say that $(\mathbf{n}_K, \varphi_K)$ is an admissible couple for $K$.

Remarks.

i) The number of vertices of $K$ is $n_1 + \cdots + n_s$

ii) The number of elements of $\sigma_i$ is $\sum_{j=1}^s a_{ij} n_j$

iii) The simplices $\sigma \in \mathcal{P}_K$ are determined by the sequence $\langle \sigma, \omega_1 \rangle, \ldots, \langle \sigma, \omega_s \rangle$, where $\langle \sigma, \omega_j \rangle$ is 1 or 0 depending on whether or not $\omega_j$ is a subset of $\sigma$.  


iv) It is clear the the couple \((n_K, \varphi_K)\) is determined by \(K\) once we have chosen an order \(\omega_1, \ldots, \omega_s\) of \(\Sigma_K\) together with an order \(\sigma_1, \ldots, \sigma_r\) of the set of maximal simplices of \(K\). Therefore \((n_K, \varphi_K)\) is determined by \(K\) up to an arbitrary permutation of rows of \(\varphi_K\) or any permutation of the components of \(n_K\) and the same permutation of the columns of \(\varphi_K\).

v) If \((n_K, \varphi_K)\) is an admissible couple for \(K\), then the corresponding couple for the simplicial complex \(\Sigma_K\) is \((n_{\Sigma}, \varphi_{\Sigma})\), with \(\varphi_{\Sigma} = \varphi_K\) and \(n_{\Sigma} = (1, \ldots, 1)\).

vi) The associated polyhedron \(|K|\) is not connected if and only if it has an admissible couple of the form \((n_K, \varphi_K)\) with \(\varphi_K = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\).

(2.3) The following properties of an admissible couple \((n_K, \varphi_K)\) for a simplicial complex \(K\) are clear:

(a) \(n_i\) are integers greater than 0.
(b) All the numbers \(a_{ij}\) are either 0 or 1.
(c) Each column contains at least an 1.
(d) No row is obtained from another row by turning some 1's into 0's.
(e) Any two columns are different.

(2.4) **Definition.** We say that a couple \((n, \varphi)\), where \(n = (n_1, \ldots, n_s)\) and \(\varphi\) is an \(r \times s\) matrix, is **admissible** if and only if satisfies the above properties and two admissible couples are said to be **equivalent** if and only if one is obtained from the other by permutation of rows or any permutation of the \(n_i\) and the same permutation of the columns of \(\varphi\).

(2.5) **Proposition.** Admissible couples of equivalent finite simplicial complexes are equivalent and the map that associates to each equivalence class of finite simplicial complexes the equivalence class of its admissible couple is a bijection with the set of equivalence classes of admissible couples.

In fact, let \(K\) and \(K'\) be equivalent finite simplicial complexes. Therefore we have a one to one map \(f : K \to K'\) sending the maximal simplices \(\sigma_1, \ldots, \sigma_r\) of \(K\) to the maximal simplices \(f(\sigma_1), \ldots, f(\sigma_r)\) of \(K'\).

Let \(\Sigma_K = \{\omega_1, \ldots, \omega_s\}\), then \(\Sigma_{K'} = \{f(\omega_1), \ldots, f(\omega_s)\}\) is the partition corresponding to \(K'\). It is then obvious that \(K\) and \(K'\) have equivalent couples.

Suppose now that \((n, \varphi)\) is admissible and consider the following finite simplicial complex with \(n = n_1 + \cdots + n_s\) vertices: \(K = \{1, \ldots, n\}\), \(\Sigma_K = \{\omega_1, \ldots, \omega_s\}\) with \(\omega_1 = \{1, \ldots, n_1\}, \ldots, \omega_s = \{n_1 + \cdots + n_{s-1} + 1, \ldots, n\}\) and maximal simplices \(\sigma_i = \bigcup_{j|a_{ij}=1} \omega_j\), \(i = 1, \ldots, r\). The number of elements of \(\sigma_i\) being \(\sum_{j=1}^s a_{ij} n_j\).

It is clear that \(K\) is a finite complex whose associated admissible matrix is the given one up to equivalence.
§3. de Rham complexes on a simplicial complex

Let $K$ be a finite simplicial complex with admissible $(n_K, \varphi_K)$ and let $R$ be any commutative integral domain with unit.

We may consider $K = \{1, \ldots, n\}$, $n = n_1 + \cdots + n_s$, $\omega_1 = \{1, \ldots, n_1\}, \ldots, \omega_s = \{n_1 + \cdots + n_{s-1} + 1, \ldots, n\}$ and the maximal simplices $\sigma_i = \bigcup_{(j|a_{ij}=1)}\omega_j$, $i = 1, \ldots, r$.

If $\sigma$ is a simplex of $K$, its j-th face, $j \in \sigma$, is the simplex $\partial^j \sigma$ obtained by deleting $j$ from $\sigma$.

Associated to each simplex $\sigma$ of $K$ we define a ring

$$R_\sigma = R[X_i]_{i \in \sigma}/(\sum_{i \in \sigma} X_i - 1)$$

and we have face maps $\partial^j : R_\sigma \rightarrow R_{\partial^j \sigma}$ given by sending $X_j$ to zero.

It is clear that each $R_\sigma$ is a polynomial ring, $\partial^j$ is surjective with kernel the ideal generated by the class of $X_j$ and the following relations hold

$$\partial^i \partial^j = \partial^j \partial^i \quad \forall \ \{i, j\} \subset \sigma$$

Define next $A^0_R(K)$ as follows: an element $f$ of $A^0_R(K)$ associates to each simplex $\sigma$ of $K$ an element $f(\sigma) \in R_\sigma$ such that $f(\partial^i \sigma) = \partial^j(f(\sigma))$ for all $j \in \sigma$.

Note that $A^0_R(K)$ has an obvious structure of $R$-algebra. It is the algebra of polynomial functions on the barycentric coordinates of $K$ with coefficients in $R$.

We have the algebraic de Rham complex on $A^0_R(K)$

$$(A^*_R(K), d) = (\Lambda_{A^0_R(K)} \Omega_R(A^0_R(K)), d)$$

and Sullivan’s de Rham complex $(\tilde{A}^*_R(K), d)$ defined as follows, see chapter 13 of [1]: an element $\Phi \in \tilde{A}^p_R(K), p \geq 0$, is a family $\{\Phi_{\sigma}\}_{\sigma \in K}$ such that $\Phi_{\sigma} \in \Lambda^p_{R_\sigma} \Omega_R(R_\sigma)$ and $\Omega(\partial^j)(\Phi_{\sigma}) = \Phi_{\partial^j \sigma}$ for each face map $\partial^j : R_\sigma \rightarrow R_{\partial^j \sigma}$, where $\Omega(\partial^j)$ is the induced map

$$\Omega(\partial^j) : \Lambda_{R_\sigma} \Omega_R(R_\sigma) \rightarrow \Lambda_{R_{\partial^j \sigma}} \Omega_R(R_{\partial^j \sigma}).$$

It is obvious that $\tilde{A}^0_R(K) = A^0_R(K)$.

We also have a natural homomorphism of commutative graded differential algebras

$$\varphi : A^*_R(K) \rightarrow \tilde{A}^*_R(K)$$

given by

$$\varphi(\Phi)(\sigma) = \Omega(\rho^K_\sigma)(\Phi) \in \Lambda^p_{R_\sigma} \Omega_R(R_\sigma)$$

for any simplex $\sigma$ of $K$ and $\Phi \in A^p_R(K)$.

Here, $\rho^K_\sigma : A^p_R(K) \rightarrow R_\sigma$ is the restriction epimorphism, $\rho^K_\sigma(f) = f(\sigma)$, and

$$\Omega(\rho^K_\sigma) : A^p_R(K) \rightarrow \Lambda^p_{R_\sigma} \Omega_R(R_\sigma)$$
is the corresponding induced map.

It is easy to find examples showing that $\varphi : A^*_R(K) \to \tilde{A}^*_R(K)$ is not injective in general, but we have the following result.

(3.1) Proposition. If $K$ is a finite simplicial complex $\varphi : A^*_R(K) \to \tilde{A}^*_R(K)$ defined above, is an epimorphism.

To prove (3.1) we need the following lemma.

(3.2) Extension lemma (see Proposition 13.8 of [1]). Let $\sigma = (1, \ldots, m)$ be an $m - 1$-simplex and suppose we are given $\Phi_i \in \Lambda^p_{R,\partial_{i\sigma}} \Omega_R(R_{\partial_{i\sigma}}), i = 1, \ldots, m$, such that

$$\Omega(\partial^i)(\Phi_j) = \Omega(\partial^i)(\Phi_i), \quad \{i, j\} \subset \{1, \ldots, m\}.$$ 

There exists then $\Phi \in \Lambda^p_{R,\sigma} \Omega_R(R_\sigma)$ such that $\Omega(\partial^i)(\Phi) = \Phi_i, i = 1, \ldots, m$.

Proof. The proof is that of Proposition 13.8 given in [1], except that one has to consider for a certain step in the proof the ring of fractions of $R_\sigma$ by the multiplicatively closed subset $\{(1 - X_\sigma)^q\}_{q \geq 0}$ and use the fact that formation of fractions commutes with exterior power and Kähler differentials. \(\square\)

(3.3) Proof of Proposition (3.1). Let $\Delta^{n-1}$ be the simplicial complex of all finite subsets of the set of vertices of $K$ Thus $K$ is a subcomplex of $\Delta^{n-1}$.

An obvious step by step procedure and induction on the dimension of the simplices, using the extension lemma (3.2), shows that the restriction map $\tilde{\rho}_K : A^*_R(\Delta^{n-1}) \to \tilde{A}^*_R(K)$ is surjective.

This implies that $\varphi$ is also surjective by the commutativity of the diagram

$$\begin{array}{ccc}
A^*_R(\Delta^{n-1}) & \xrightarrow{\rho_K} & A^*_R(K) \\
\downarrow{\tilde{\rho}_K} & & \downarrow{\varphi} \\
\tilde{A}^*_R(K) & & \\
\end{array}$$

where $\rho_K$ is induced by the restriction $A^0_R(\Delta^n) \to A^0_R(K)$.

(3.4) Proposition. The kernel of $\rho_K : A^0_R(\Delta^{n-1}) \to A^0_R(K)$ is generated by the set of elements $\prod_{i \in \sigma} \bar{X}_i$ for all simplices $\sigma$ of $\Delta^{n-1}$ not being simplices of $K$.

Proof. If $\sigma$ is a simplex of $\Delta^{n-1}$ we write $X_\sigma = \prod_{i \in \sigma} X_i$ and denote by $\bar{X}_\sigma = \prod_{i \in \sigma} \bar{X}_i$ its class in $A^0_R(\Delta^{n-1})$.

It is clear that the elements $\bar{X}_\sigma$, for all simplices $\sigma$ of $\Delta^{n-1}$ that are not simplices of $K$, belong to the kernel of $\rho_K : A^0_R(\Delta^{n-1}) \to A^0_R(K)$.

It remains to be proved that the above elements $\bar{X}_\sigma$ generate $\ker\rho_K$ and this is done by induction on the number of maximal simplices of $K$. \(\square\)

(3.5) Corollary. The algebra $A^0_R(K)$ is obtained directly from the admissible couple $(n_K, \varphi_K)$, up to isomorphism, as follows:

$$A^0_R(K) = R[X_1, \ldots, X_n]/(X_1 + \cdots + X_n - 1, (X_i^\alpha)_{\alpha, t})$$
where \( n = n_1 + \cdots + n_s \), \( X_i^\alpha = X_{t_1}^{\alpha_1} \cdots X_{t_s}^{\alpha_s} \), \( \alpha = \alpha_1, \ldots, \alpha_s \) is a sequence with each \( \alpha_i \) being either 0 or 1, \( 1 \leq t_1 \leq n_1; \ldots; n_1 + \ldots n_{s-1} + 1 \leq t_s \leq n \), and \( \alpha \) and \( t \) satisfy the following two conditions:

a) For all \( i \in \{1, \ldots, r\} \) there exists \( \rho(i) \in \{1, \ldots, s\} \) such that \( \alpha_{\rho(i)} = 1 \) but \( a_{i,\rho(i)} = 0 \)

b) If \( \alpha_j = 1 \) there exists \( i \in \{1, \ldots, r\} \) such that \( a_{ik} = 0, \ k \neq j \Rightarrow \alpha_k = 0 \)

**Proof.** For each \( \alpha, t \) as above, define a simplex \( \sigma \) of \( \Delta^{n-1} \) by

\[
\sigma \cap \omega_k = \begin{cases} \{t_k\} & \text{if } \alpha_k = 1 \\ \emptyset & \text{if } \alpha_k = 0 \end{cases}
\]

Condition (a) tells us that \( \sigma \) is not contained in any of the maximal simplices \( \sigma_i \) of \( K \), i.e. \( \sigma \) is not a simplex of \( K \), and condition (b) says that \( \sigma \) is minimal among the simplices of \( \Delta^{n-1} \) that are not simplices of \( K \), i.e. we obtain a simplex of \( K \) by deleting any vertex of \( \sigma \).

The proof of this corollary is now an obvious consequence of our previous proposition. \( \square \)

(3.6) **Theorem.** Let \( K \) be a finite simplicial complex, \( \sigma_1, \ldots, \sigma_r \) the maximal simplices of \( K \) and \( p_i, i = 1, \ldots, r \) the kernels of the restriction epimorphisms \( \rho^K_{\sigma_i} : A^0_R(K) \to R_{\sigma_i} \). Then :

a) For any homomorphism of algebras \( \mu : A^0_R(K) \to P \), \( P \) being any polynomial algebra on \( R \) with a finite number of variables, there exists \( i \in \{1, \ldots, r\} \) and a homomorphism of algebras \( \mu_i : A^0_R(K)/p_i \to P \) such that \( \mu = \mu_i \circ \pi_i \), where \( \pi_i : A^0_R(K) \to A^0_R(K)/p_i \) is the canonical projection.

b) \( p_1, \ldots, p_r \) is the set of minimal ideals of \( A^0_R(K) \) having the property that the quotients \( A^0_R(K)/p_i \) are polynomial algebras on \( R \) with a finite number of variables.

c) For all \( I \subset \{1, \ldots, r\} \), \( \sum_{i \in I} p_i = \ker \rho^K_{\sigma_I} \) if \( \sigma_I = \bigcap_{i \in I} \sigma_i \neq \emptyset \) and \( \sum_{i \in I} p_i = A^0_R(K) \) if \( \bigcap_{i \in I} \sigma_i = \emptyset \).

d) For any couple of subsets \( I, J \) of \( \{1, \ldots, r\} \), \( \sum_{i \in I} p_i = \sum_{j \in J} p_j \) if and only if \( \bigcap_{i \in I} \sigma_i = \bigcap_{j \in J} \sigma_j \).

Observe that c) and d) establishes a one to one correspondence from the set of simplices of \( K \) appearing as intersections of maximal simplices \( \sigma_1, \ldots, \sigma_r \) and the set of ideals of \( A^0_R(K) \) that are sums of minimal ideals \( p_1, \ldots, p_r \).

**Proof.** a) Let \( \Delta^{n-1} \) be the simplicial complex of all finite subsets of \( K^0 \) and consider the algebra homomorphism

\[
\mu \circ \rho_K : A^0_R(\Delta^{n-1}) = R[X_1, \ldots, X_n]/(\sum_{i=1}^n X_i - 1) \to P.
\]

Define a simplex \( \sigma \) of \( \Delta^{n-1} \) by \( i \in \sigma \Leftrightarrow \mu(\rho_K(X_i)) \neq 0 \)
Observe that $\sigma$ is not empty: otherwise, $\mu(\rho_K(\bar{X}_i)) = 0$, $i = 1, \ldots, n$ and so
\[
0 = \mu(\rho_K(\sum_{i=1}^{n} \bar{X}_i)) = \mu(\rho_K(1)) = 1,
\]
which is a contradiction.

Moreover, $\sigma$ is a simplex of $K$. In fact, if $\sigma$ were not so, Proposition (3.4) would imply that $\rho_K(\prod_{i \in \sigma} \bar{X}_i) = 0$. But then $\prod_{i \in \sigma} \mu(\rho_K(\bar{X}_i)) = 0$ and, since $R$ has no zero-divisors, we would have $\mu(\rho_K(\bar{X}_i)) = 0$ for some $i \in \sigma$, which contradicts the definition of $\sigma$.

Consider then the unique algebra homomorphism $\bar{\mu}$ making commutative the following diagram
\[
\begin{array}{ccc}
A_R^0(\Delta^{n-1}) & \xrightarrow{\rho_\sigma} & R_\sigma \\
\rho_K & \downarrow & \downarrow \bar{\mu} \\
A_R^0(K) & \xrightarrow{\mu} & P
\end{array}
\]
Note that $\bar{\mu}$ is well defined because, by definition of $\sigma$,
\[
\mu(\rho_K(\sum_{i \in \sigma} \bar{X}_i)) = \mu(\rho_K(\sum_{i=1}^{n} \bar{X}_i)) = 1.
\]

On the other hand the following diagram commutes
\[
\begin{array}{ccc}
A_R^0(K) & \xrightarrow{\rho^K_\sigma} & R_\sigma \\
\mu & \swarrow & \downarrow \bar{\mu} \\
P & \downarrow & \\
\end{array}
\]
In fact,
\[
\bar{\mu} \circ \rho^K_\sigma \circ \rho_K = \bar{\mu} \circ \rho_\sigma = \mu \circ \rho_K
\]
and since $\rho_K : A_R^0(\Delta^{n-1}) \to A_R^0(K)$ is surjective, we have $\bar{\mu} \circ \rho^K_\sigma = \mu$.

Choose now any maximal simplex $\sigma_i$ of $K$ such that $\sigma \subset \sigma_i$ and let $\mu_i$ be the composite
\[
A_R^0(K)/p_i \xrightarrow{\cong} R_{\sigma_i} \longrightarrow R_\sigma \bar{\mu} \rightarrow P
\]
Clearly we have $\mu_i \circ \pi_i = \mu$ as desired.

b) Let $p$ be any ideal of $A_R^0(K)$ such that the quotient $A_R^0(K)/p$ is isomorphic to some polynomial algebra on $R$ with a finite number of variables.

By (a), there exist $j \in \{1, \ldots, r\}$ and $\mu_j : A_R^0(K)/p_j \rightarrow A_R^0(K)/p$, homomorphism of algebras, such that the projection $A_R^0(K) \rightarrow A_R^0(K)/p$ is the composite
\[
A_R^0(K) \xrightarrow{\rho^K_j} A_R^0(K)/p_j \xrightarrow{\mu_j} A_R^0(K)/p
\]
This shows that $p_j \subset p$. 
On the other hand if \( p_i \subset p_j \) for \( i, j \in \{1, \ldots, r\} \), we have then \( \sigma_j \subset \sigma_i \). In fact if there is a \( k \in \sigma_j - \sigma_i \) we have
\[
\rho^K_{\sigma_i}(\rho_K(\bar{X}_k)) = \rho_{\sigma_i}(\bar{X}_k) = 0,
\]
but
\[
\rho^K_{\sigma_j}(\rho_K(\bar{X}_k)) = \rho_{\sigma_j}(\bar{X}_k) \neq 0.
\]
Therefore \( \rho_K(\bar{X}_k) \in p_i \) and \( \rho_K(\bar{X}_k) \notin p_j \), which contradicts the hypothesis \( p_i \subset p_j \).

Thus \( \sigma_i \subset \sigma_j \), and since \( \sigma_i \) and \( \sigma_j \) are maximal simplices of \( K \), we have \( \sigma_i = \sigma_j \) and so \( p_i = p_j \).

c) For any simplex \( \sigma \) of \( K \) we have \( \ker \rho^K_{\sigma} = \rho_K(\ker \rho_{\sigma}) \) because of the relation \( \rho^K_{\sigma} \circ \rho_K = \rho_{\sigma} \) and the surjectivity of \( \rho_K \).

According to Proposition (3.4), the set of elements \( \bar{X}_i \in A^0_R(\Delta^{n-1}) \), for all \( i \notin \sigma \), generate \( \ker \rho_{\sigma} \) and so the elements \( \rho_K(\bar{X}_i) \in A^0_R(K) \), for all \( i \notin \sigma \), generate \( \ker \rho^K_{\sigma} \).

If we take now \( \sigma = \sigma_I = \bigcap_{i \in I} \sigma_i \) for some \( I \subset \{1, \ldots, r\} \) and we assume \( \sigma_I \neq \emptyset \), then we have that \( \ker \rho^K_{\sigma_I} \) is generated by the elements \( \rho_K(\bar{X}_i) \) for all \( i \notin \sigma_I \), i.e.
\[
\ker \rho^K_{\sigma_I} = \sum_{i \notin I} \ker \rho^K_{\sigma_i} = \sum_{i \notin I} p_i.
\]

Finally, if \( \bigcap_{i \in I} \sigma_i = \emptyset \), for any \( i \in \{1, \ldots, n\} \) there exists \( j \in I \) such that \( i \notin \sigma_j \).
Therefore \( \rho_K(\bar{X}_i) \in \ker \rho^K_{\sigma_j} = p_j \) and so \( \sum_{i \in I} p_i = A^0_R(K) \).

d) If \( \sigma_I = \sigma_J \), then \( \ker \rho^K_{\sigma_I} = \ker \rho^K_{\sigma_J} \) and so \( \sum_{i \in I} p_i = \ker \rho^K_{\sigma_I} = \ker \rho^K_{\sigma_J} = \sum_{j \in J} p_j \).

Conversely if \( \sum_{i \in I} p_i = \sum_{j \in J} p_j \), then \( \ker \rho^K_{\sigma_I} = \ker \rho^K_{\sigma_J} \). Assume \( k \) is an element of \( \sigma_I - \sigma_J \), then \( \rho_K(\bar{X}_k) \notin \ker \rho^K_{\sigma_I} \). However \( \rho_K(\bar{X}_k) \in \ker \rho^K_{\sigma_J} \), which is a contradiction. Hence \( \sigma_I = \sigma_J \). \( \square \)

(3.7) Corollary. For any finite simplicial complex \( K \) and any ring \( R \) the algebra of Sullivan 0-forms \( A^\ast_R(K) \) determines the Sullivan’s de Rham complex \( \check{A}^\ast_R(K) \).

Proof. Consider the epimorphism \( \varphi : A^\ast_R(K) \to \check{A}^\ast_R(K) \) and we have to show that its kernel can be deduced directly from \( A^0_R(K) \). \( \square \)

Let \( p_1, \ldots, p_r \) be the set of minimal ideals of \( A^0_R(K) \) having the property that the quotients \( A^0_R(K)/p_i \) are polynomial algebras on \( R \) with a finite number of variables. This set is completely determined by the algebra \( A^0_R(K) \).

But Theorem (3.6) says, in particular, that \( \{p_1, \ldots, p_r\} = \{\ker \rho^K_{\sigma_1}, \ldots, \ker \rho^K_{\sigma_s}\} \), where \( \sigma_1, \ldots, \sigma_s \) are the maximal simplices of \( K \).

Therefore we have
\[
\ker \varphi = \cap_{i=1}^s \ker \Omega(\rho^K_{\sigma_i}) = \cap_{i=1}^s \ker \Omega(\pi_i),
\]
where \( \pi_i : A^0_R(K) \to A^0_R(K)/p_i \) are the canonical projections and \( \Omega(\pi_i) \) the corresponding induced maps for the algebraic de Rham complexes.

It is interesting to observe, as the following example shows, that \( \varphi \) does not induce in general an isomorphism in cohomology. Therefore, to compute the cohomology of \( \tilde{A}^*_{R}(K) \), which for \( R \) being a field of characteristic zero is the “correct” cohomology of \( K \), one cannot simply compute the cohomology of the algebraic De Rham complex of \( A^0_R(K) \).

(3.8) Example. Let \( K \) be the simplicial complex having three 0-simplices 1, 2, 3 and three 1-simplices 12, 23, 31. The geometric realization of \( K \) is, of course, \( S^1 \).

\[ A^0_R(K) \text{ is the quotient of the polynomial ring } R[X_1, X_2, X_3] \text{ by the ideal generated by } X_1X_2X_3, X_1 + X_2 + X_3 - 1. \]

Then
\[
\varphi^* : H^1(A^*_{R}(K)) \to H^1(\tilde{A}^*_{R}(K))
\]
is not an isomorphism.

In fact, one checks easily that \( \Phi = \bar{X}_1^2 \bar{X}_2^2 d\bar{X}_3 \) satisfies: \( \Phi \in \ker \varphi \), \( d\Phi = 0 \) and \( \Phi \neq 0 \). In particular, \( \Phi \) represents a cohomology class in \( H^1(A^*_{R}(K)) \) which applies to 0 in \( H^1(\tilde{A}^*_{R}(K)) \).

However, if we had \( \Phi = df \) for some \( f \in A^0_R(K) \) then one deduces by applying \( \varphi \) that \( \tilde{d}f = 0 \), where \( \tilde{d} \) denotes the differential in \( \tilde{A}^*_{R}(K) \), and so \( f \in R \) and therefore \( \Phi = df = 0 \), which is not true.

(3.9) Theorem. A finite simplicial complex \( K \) is determined up to simplicial equivalence either by its associated admissible couple \( (n_K, \varphi_K) \) or by its algebra of polynomial functions with coefficient \( R \) on the baricentric coordinates of \( K \).

Proof. Observe that as a consequence of Theorem (3.6) we deduce from the algebra \( A = A^0_R(K) \) the set of minimal ideals \( p_1, \ldots, p_r \), with respect to the property that the quotients \( A/p_i \) are polynomial algebras and so we have the set \( P_A \) of ideals appearing as sum of some of the ideals \( p_1, \ldots, p_r \) and in particular for such ideals \( p \) we know the number \( n(p) - 1 \) of variables of the polynomial algebra \( A/p \).

For each \( p \in P_A \) use (2.1) to define the number
\[
n_p = \sum_{q} (-1)^{n(q)} n(q)
\]
where \( q \) in the sum runs through the members of \( P_A \) of the form \( p + p_i \), for \( i = 1, \ldots, r \).

Let \( q_1, \ldots, q_s \) be all the members of \( P_A \) such that \( n_{q_i} > 0 \) and define numbers \( n_i = n_{q_i}, i = 1, \ldots, s \) and \( a_{ij} = 1 \) or 0 depending on whether \( p_i \subset q_j \) or \( p_i \not\subset q_j \).

Therefore we obtain an admissible couple \( (n, \varphi) \) which clearly coincides with \( (n_K, \varphi_K) \).

Finally we use Proposition (2.5) to complete the proof of our theorem. \( \square \)
REFERENCES


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