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COMMON FIXED POINTS OF GREGUŠ TYPE MULTI–VALUED MAPPINGS

R. A. RASHWAN AND M. A. AHMED

Abstract. This work is considered as a continuation of [19,20,24]. The concepts of \( \delta \)-compatibility and sub-compatibility of Li-Shan [19, 20] between a set-valued mapping and a single-valued mapping are used to establish some common fixed point theorems of Greguš type under a \( \phi \)-type contraction on convex metric spaces. Extensions of known results, especially theorems by Fisher and Sessa [11] (Theorem B below) and Jungck [16] are thereby obtained. An example is given to support our extension.

1. Introduction

Fixed point theory of single-valued and multi-valued maps has been investigated extensively and applied to diverse problems during the last few decades. This theory provides techniques for solving a variety of applied problems in mathematical science and engineering (see e.g., [1, 2, 3, 23]).

In 1970, Takahashi [28] introduced a notion of convexity in metric spaces (see Definition 2.7) and generalized some fixed point theorems in Banach spaces. Subsequently, Ciric [6, 7], Gauy, Singh and Whitfield [14] and others have studied convex metric spaces and fixed point theorems.

In [13], Greguš proved the following theorem:

Theorem A. Let \( C \) be a nonempty closed convex subset of a Banach space \( X \) and \( T \) be a mapping of \( C \) into itself satisfying the inequality

\[
\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Ty - y\|,
\]

for all \( x, y \) in \( C \), where \( a > 0, b \geq 0, c \geq 0 \) and \( a + b + c = 1 \). Then \( T \) has a unique fixed point.

Fisher and Sessa [11] established a generalization of Theorem A as follows:

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**Theorem B.** Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $T, f$ be two weakly commuting mappings of $C$ into itself satisfying the inequality
\[ \|Tx - Ty\| \leq a\|fx - fy\| + (1 - a) \max\{\|Tx - fx\|, \|Ty - fy\|\}, \]
for all $x, y \in C$, where $0 < a < 1$. If $f$ is linear and nonexpansive in $C$ such that $fC$ contains $TC$, then $T$ and $f$ have a unique common fixed point in $C$.

In recent years, common fixed points of Greguš type have been obtained by Ciric [4, 5], Davies and Sessa [8], Diviccaro, Fisher and Sessa [9], Jungck [16], Khan and Imdad [18], Murthy, Cho and Fisher [22] and Sessa and Fisher [26] in Banach spaces. On the other hand, Jungck [16] and Mukherjee and Verma [21] replaced linearity and nonexpansiveness by affine and continuity mappings, respectively. In [8, 22], the authors replaced nonexpansiveness, linearity and weak commutativity by continuity and compatibility. Also, Many theorems which are closely related to Greguš Theorem extended to multivalued mappings such as Li-Shan [19, 20] and Rashwan and Ahmed [24].

The aim of this paper is to prove some common fixed point theorems of Greguš type under a $\phi$-contraction. Our results extend Theorems A, B and Jungck [16] to multi-valued mappings.

2. Basic Preliminaries

In the sequel, $(X, d)$ denotes a metric space and $B(X)$ is the set of all nonempty bounded subsets of $X$. As in [10, 12], we define
\[ \delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}, \]
for all $A, B$ in $B(X)$. If $A$ consists of a single point $a$, we write $\delta(A, B) = \delta(a, B)$. Also, if $B$ contains a single point $b$, it yields that $\delta(A, B) = d(a, b)$.

It follows immediately from the definition of $\delta(A, B)$ that
\[
\begin{align*}
\delta(A, B) &= \delta(B, A) \geq 0, \\
\delta(A, B) &\leq \delta(A, C) + \delta(C, B), \\
\delta(A, B) &= 0 \text{ iff } A = B = \{a\}, \\
\delta(A, A) &= \text{diam}\ A,
\end{align*}
\]
for all $A, B, C \in B(X)$.

**Definition 1.1** [10]. A sequence $\{A_n\}$ of nonempty subsets of $X$ is said to be convergent to a subset $A$ of $X$ if:

(i) each point $a$ in $A$ is the limit of a convergent sequence $\{a_n\}$, where $a_n$ is in $A_n$ for $n \in N$ ($N$ : the set of all positive integers),

(ii) for arbitrary $\epsilon > 0$, there exists an integer $m$ such that $A_n \subseteq A_\epsilon$ for $n > m$, where $A_\epsilon$ denotes the set of all points $x$ in $X$ for which there exists a point $a$ in $A$, depending on $x$, such that $d(x, a) < \epsilon$.

$A$ is then said to be the limit of the sequence $\{A_n\}$. 
**Lemma 2.1** [10]. If \( \{A_n\} \) and \( \{B_n\} \) are sequences in \( B(X) \) converging to \( A \) and \( B \) in \( B(X) \), respectively, then the sequence \( \{\delta(A_n, B_n)\} \) converges to \( \delta(A, B) \).

**Lemma 2.2** [12]. Let \( \{A_n\} \) be a sequence in \( B(X) \) and \( y \) be a point in \( X \) such that \( \delta(A_n, y) \to 0 \). Then the sequence \( \{A_n\} \) converges to the set \( \{y\} \) in \( B(X) \).

**Definition 2.2** [12]. A set-valued mapping \( F \) of \( X \) into \( B(X) \) is said to be continuous at \( x \in X \) if the sequence \( \{Fx_n\} \) in \( B(X) \) converges to \( Fx \) whenever \( \{x_n\} \) is a sequence in \( X \) converging to \( x \) in \( X \). \( F \) is said to be continuous on \( X \) if it is continuous at every point in \( X \).

**Lemma 2.3** [12]. Let \( \{A_n\} \) be a sequence of nonempty subsets of \( X \) and \( z \) be in \( X \) such that \( \lim_{n \to \infty} a_n = z \), \( z \) being independent of the particular choice of each \( a_n \in A_n \). If a selfmap \( f \) of \( X \) is continuous, then \( \{fx\} \) is the limit of the sequence \( \{fA_n\} \).

**Definition 2.3** [27]. The mappings \( F : X \to B(X) \) and \( f : X \to X \) are said to be weakly commuting if \( Ffx \in B(X) \) and

\[
\delta(Ffx, fFx) \leq \max\{\delta(fx, Fx), \text{diam } Fx\},
\]

for all \( x \) in \( X \).

Note that if \( F \) is a single-valued mapping, then the set \( \{Ffx\} \) consists of a single point. Therefore, \( \text{diam } Fx = 0 \) for all \( x \in X \) and condition (2.1) reduces to the condition given by Sessa [25], that is

\[
d(Ffx, FFx) \leq d(fx, Fx),
\]

for all \( x \) in \( X \).

Two commuting mappings \( F \) and \( f \) clearly weakly commute but two weakly commuting \( F \) and \( f \) do not necessarily commute as shown in [27].

In [15], Jungck generalized the concept of weakly commuting for single-valued mappings in the following way:

**Definition 2.4.** Two single-valued mappings \( f \) and \( g \) of a metric space \( (X, d) \) into itself are compatible if \( \lim_{n \to \infty} d(fgx_n, gfx_n) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} gx_n = t \) for some \( t \) in \( X \).

It can be seen that two weakly commuting mappings are compatible but the converse is false. Examples supporting this fact can be found in [15].

In [19], Li-Shan extended the definition 2.4 of compatibility to set-valued mappings as follows:

**Definition 2.5.** The mappings \( f : X \to X \) and \( F : X \to B(X) \) are \( \delta \)-compatible if \( \lim_{n \to \infty} \delta(Ffx_n, FFx_n) = 0 \) whenever \( \{x_n\} \) is a sequence in \( X \) such that \( Ffx_n \in B(X) \), \( Fx_n \to \{t\} \) and \( fx_n \to t \) for some \( t \) in \( X \).
Definition 2.6. The mappings \( f : X \to X \) and \( F : X \to B(X) \) are subcompatible if \( \{ t \in X : Ft = \{ ft \} \} \subseteq \{ t \in X : Fft = fFt \} \).

Remark 2.1. In [19], Li-Shan pointed out that the pair \( \{ F, f \} \) is \( \delta \)-compatible \( \iff (F, f) \) is subcompatible but the converse is not true.

The following proposition of Jungck and Rhoades [17] is useful in the sequel:

Proposition 2.1. Let \( (X, d) \) be a complete metric space. Suppose that \( f : X \to X \) and \( F : X \to B(X) \) and the pair \( \{ F, f \} \) is \( \delta \)-compatible.

\( (P_1) \) Suppose that the sequences \( \{ fx_n \} \) and \( \{ Fx_n \} \) converge to \( t \in X \) and \( \{ t \} \), respectively. If \( f \) is continuous, then \( Ffx_n \to \{ ft \} \).

\( (P_2) \) If \( \{ ft \} = Ft \) for some \( t \in X \), then \( Fft = fFt \).

Now, we need some definitions due to Takahashi [28]:

Definition 2.7. Let \( X \) be a metric space and \( I = [0, 1] \) be the closed unit interval. A continuous mapping \( W : X \times X \times I \to X \) is said to be a convex structure on \( X \) if there is \( \lambda \in I \) such that for all \( x, y, u \in X \)

\[
d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).
\]

\( X \) together with a convex structure is called a convex metric space.

Clearly, a Banach space or any convex subset of it is a convex metric space with \( W(x, y, \lambda) = \lambda x + (1 - \lambda)y \). More generally, if \( X \) is a linear space with a translation invariant metric satisfying

\[
d(\lambda x + (1 - \lambda)y, 0) \leq \lambda d(x, 0) + (1 - \lambda)d(y, 0),
\]

then \( X \) is a convex metric space.

Definition 2.8. Let \( X \) be a convex metric space. A nonempty subset \( K \) of \( X \) is convex if \( W(x, y, \lambda) \in K \) whenever \( x, y \in K \) and \( \lambda \in I \).

Throughout this paper, a convex metric space will be denoted by \( (X, d, W) \). Let \( \Phi \) be the set of all functions \( \phi : [0, \infty) \to [0, \infty) \) which satisfies the following conditions:

\( (i) \) \( \phi \) is non-decreasing and continuous from the right,

\( (ii) \) \( \phi(t) < t \) for each \( t > 0 \).

Let \( F : X \to B(X), f : X \to X \) be mappings on a metric space \( X \) satisfying the following inequality:

\[
\delta(Fx, Fy) \leq \phi(ad(fx, fy) + (1 - a) \max\{\delta(Fx, fx), \delta(Fy, fy)\}),
\]

for all \( x, y \in X \), where \( 0 < a < 1 \) and \( \phi \in \Phi \).

For our main theorems we need the following lemma, its proof is similar to that of Lemma 2.3 in [20]:

Lemma 2.4. Let \( K \) be a nonempty closed subset of a complete metric space \( (X, d) \). If the mappings \( f : K \to K \) and \( F : K \to B(K) \) satisfy the condition (2.3), then

(\( I \)) \( F \) and \( f \) have at most one common fixed point \( u \) in \( K \) and further \( Fu = \{ u \} \);

(II) if \( \{ x_n \} \) is a sequence in \( K \) such that \( \delta(Fx_n, fx_n) \to 0 \), then there exists a \( u \in K \) such that \( Fx_n \to \{ u \} \) and \( fx_n \to u \).
3. Main Results

The following theorem is useful in proving Theorem 3.2:

**Theorem 3.1.** Let $K$ be a nonempty closed subset of a complete metric space $(X,d)$. Furthermore, let $F : K \to B(K)$ and $f : K \to K$ be a multivalued mapping and a single-valued mapping, respectively satisfying the inequality (2.3).

1. If $F$ and $f$ have a unique common fixed point $u$ in $K$ and $Fu = \{u\}$, then
\[
\inf \{\delta(Fx, fx) : x \in K\} = 0.
\]

2. If $\inf \{\delta(Fx, fx) : x \in K\} = 0$ and $F, f$ satisfy one of the following conditions:
   
   (U) the pair $\{F, f\}$ is $\delta$-compatible and $f$ is continuous;
   
   (V) the pair $\{F, f\}$ is $\delta$-compatible, $FK \subseteq fK$ and $F$ is continuous;
   
   (Z) the pair $\{F, f\}$ is subcompatible and $f$ is surjective,

then $F$ and $f$ have a unique common fixed point $u$ in $K$ and $Fu = \{u\}$.

**Proof.** (1) Suppose that $u$ is a unique common fixed point of $F$ and $f$ in $K$. Using the inequality (2.3), we obtain that
\[
\delta(Fu, u) \leq \delta(Fu, Fu) \leq \phi((1 - a)\delta(Fu, u)) < \delta(Fu, u).
\]

This contradiction implies that $Fu = \{u\}$. So, $\inf \{\delta(Fx, fx) : x \in K\} = 0$. To prove (2) let $\{x_n\}$ be a sequence such that
\[
\delta(Fx_n, fx_n) \to \inf \{\delta(Fx, fx) : x \in K\} = 0.
\]

By Lemma 2.4 (II), there exists a point $u \in K$ such that the sequences $\{fx_n\}$ and $\{Fx_n\}$ converge to $u$ and $\{u\}$, respectively.

Now suppose that (U) holds. Since $f$ is continuous, then Lemma 2.3 shows that the sequences $\{fx_n\}$ and $\{Fx_n\}$ converge to $fu$ and $\{fu\}$, respectively. Proposition 2.1 ($P_1$) implies that the sequence $\{Ffx_n\}$ converges to $\{fu\}$. Applying the inequality (2.3), we get that
\[
\delta(Ffx_n, Fx_n) \leq \phi(ad(f^2x_n, fx_n) + (1 - a)\max\{\delta(f^2x_n, Ffx_n), \delta(Fx_n, fx_n)\}).
\]

Letting $n \to \infty$, it implies from Lemma 2.1 that
\[
d(fu, u) \leq \phi(ad(fu, u)) < f(fu, u) < d(fu, u).
\]

This contradiction demands that $fu = u$. From the inequality (2.3), it yields that
\[
\delta(Fx_n, Fu) \leq \phi(ad(fx_n, fu) + (1 - a)\max\{\delta(fx_n, fx_n), \delta(Fu, fu)\}).
\]

Letting $n \to \infty$, it follows from Lemma 2.1 that
\[
\delta(Fu, u) \leq \phi((1 - a)\delta(Fu, u)) < \delta(u, Fu).
\]

This contradiction follows that $Fu = \{u\}$. Therefore, we know from Lemma 2.4 (I) that $u$ is the unique common fixed point of $F$ and $f$ and $Fu = \{u\}$.
Now suppose that (V) holds. Then the sequence \( \{ Ff x_n \} \) converges to \( Fu \). Let \( u_n \) be an arbitrary point in \( F x_n \) for \( n = 1, 2, \ldots \). Since \( d(u_n, u) \leq \delta(F x_n, u) \) and \( F \) is continuous, then we get that the sequence \( \{ F u_n \} \) converges to \( Fu \). By the inequality (2.3), we deduce that
\[
\delta(Fu_n, Fu_n) \leq \phi((1 - a)\delta(Fu_n, fu_n)) \\
\leq \phi((1 - a)[\delta(Fu_n, Ff x_n) + \delta(F f x_n, f F x_n)]).
\]
As \( n \to \infty \), the \( \delta \)-compatibility of \( \{ F, f \} \) and Lemma 2.1 lead to
\[
\delta(Fu, Fu) \leq \phi((1 - a)\delta(Fu, Fu)) < \delta(Fu, Fu).
\]
This contradiction gives that \( \delta(Fu, Fu) = 0 \). From the inequality (2.3), we obtain that
\[
\delta(Fu_n, F x_n) \leq \phi(a d(fu_n, f x_n) + (1 - a) \max\{\delta(Fu_n, fu_n), \delta(F x_n, f x_n)\}) \\
\leq \phi(a [\delta(F f x_n, F f x_n) + \delta(F f x_n, f x_n)] \\
+ (1 - a) \max\{\delta(Fu_n, F f x_n) + \delta(F f x_n, f F x_n), \delta(F x_n, f x_n)\}).
\]
Since \( \phi \) is continuous from the right and the pair \( \{ F, f \} \) is \( \delta \)-compatible, as \( n \to \infty \), using Lemma 2.1, we have that
\[
\delta(Fu, u) \leq \phi(a \delta(F, u) + (1 - a)\delta(Fu, Fu)) < a\delta(Fu, u) < \delta(Fu, u).
\]
This implies that \( Fu = \{ u \} \). Since \( FK \subseteq fK \), then there exists a point \( w \) in \( K \) such that \( fw = u \), it yields from inequality (2.3) that
\[
\delta(F x_n, Fw) \leq \phi(ad(f x_n, fw) + (1 - a) \max\{\delta(F x_n, f x_n), \delta(Fw, fw)\}).
\]
Letting \( n \to \infty \), the last inequality becomes
\[
\delta(u, Fw) \leq \phi((1 - a)\delta(F, u)) < \delta(F, u).
\]
This contradiction implies that \( Fw = \{ u \} \). Since \( \{ F, f \} \) is \( \delta \)-compatible and \( \{ fw \} = Fw \) for some \( w \in K \), then Proposition 2.1 \( (P_2) \) leads to
\[
\{ u \} = Fu = Ff w = f F w = \{ fu \}.
\]
It follows from Lemma 2.4 (I) that \( u \) is the unique common fixed point of \( F \) and \( f \) and \( Fu = \{ u \} \).

Now suppose that (Z) holds. Then there exists a point \( v \) in \( K \) such that \( fv = u \). From the inequality (2.3), we obtain that
\[
\delta(Fv, F x_n) \leq \phi(ad(fv, f x_n) + (1 - a) \max\{\delta(Fv, fv), \delta(F x_n, f x_n)\}).
\]
Letting \( n \to \infty \), we get from Lemma 2.1 that
\[
\delta(Fv, u) \leq \phi((1-a)\delta(Fv, u)) < \delta(Fv, u).
\]
This contradiction implies that \( Fv = \{u\} \). Since \( \{F, f\} \) is subcompatible, we have that \( Fu = Ffv = fFv = \{fu\} \). Using again the inequality (2.3), we deduce that
\[
\delta(Fu, Fx_n) \leq \phi(ad(fu, fx_n) + (1-a)\max\{\delta(Fu, fu), \delta(Fx_n, fx_n)\}).
\]
As \( n \to \infty \), Lemma 2.1 gives that \( d(fu, u) \leq \phi(ad(fu, u)) < ad(fu, u) < d(fu, u) \).

It follows that \( fu = u \). From Lemma 2.4 (I), \( u \) is the unique common fixed point of \( F \) and \( Fu = \{u\} \).

Now, we are ready to prove the following theorem:

**Theorem 3.2.** Let \( K \) be a nonempty closed subset of a complete convex metric space \((X, d, W)\) and \( F : K \to B(K) \), \( f : K \to K \) be mappings satisfying the inequality (2.3). If \( fK \) is a convex subset of \( X \) such that \( FK \subseteq fK \) and \( F, f \) satisfy one of the three conditions in Theorem 3.1, then \( F \) and \( f \) have a unique common fixed point \( u \) in \( K \) and \( Fu = \{u\} \).

**Proof.** Let \( x_0 \) be an arbitrary point in \( K \). Since \( FK \subseteq fK \), we choose points \( x_1, x_2, x_3 \) in \( K \) such that \( fx_1 \in Fx, fx_2 \in Fx_1, fx_3 \in Fx_2 \). For \( i = 1, 2, 3 \), we obtain from the inequality (2.3) that
\[
\delta(Fx_i, fx_i) \leq \delta(Fx_i, Fx_{i-1})
\]
\[
\leq \phi(ad(fx_i, fx_{i-1}) + (1-a)\max\{\delta(Fx_i, fx_i), \delta(Fx_{i-1}, fx_{i-1})\})
\]
\[
\leq \phi(ad(fx_i, fx_{i-1}) + (1-a)\max\{\delta(Fx_i, fx_i), \delta(Fx_{i-1}, fx_{i-1})\})
\]
If \( \delta(Fx_i, fx_i) \geq \delta(Fx_{i-1}, fx_{i-1}) \), then
\[
\delta(Fx_i, fx_i) \leq \phi(\delta(Fx_i, Fx_i)) < \delta(Fx_i, fx_i)
\]
This contradiction implies that
\[
\delta(Fx_i, fx_i) < \delta(Fx_{i-1}, fx_{i-1})
\]
for \( i = 1, 2, 3 \). It follows that
\[
\delta(Fx_i, fx_i) < \delta(Fx_0, f0)
\]
for \( i = 1, 2, 3 \). Since \( fK \) is convex, then there exists \( w \) in \( K \) such that
\[
fw = W(fx_2, fx_3, \tfrac{1}{2}) \in W(Fx_1, Fx_2, \tfrac{1}{2})
\]
where \( W(Fx_1, Fx_2, \frac{1}{2}) = \cup \{ W(e, m, \frac{1}{2}) : e \in Fx_1, m \in Fx_2 \}. \quad \square \)

Using the inequalities (2.3) and (3.1), we have from the definition of convex structure that

\[
d(fx_1, fw) \leq \delta(fx_1, W(Fx_1, Fx_2, \frac{1}{2})) \\
\leq \frac{1}{2} [\delta(fx_1, Fx_1) + \delta(fx_1, Fx_2)] \\
\leq \frac{1}{2} [\delta(fx_1, Fx_1) + \delta(Fx_0, Fx_2)] \\
< \frac{1}{2} [\delta(fx_0, Fx_0) + \phi(ad(fx_0, fx_2) \\
+ (1 - a) \max \{\delta(fx_0, Fx_0), \delta(Fx_2, fx_2)\})] \\
< \frac{a + 2}{2} \delta(fx_0, fx_0).
\]

(3.2)

Also, we have from the inequality (3.1) and the definition of convex structure that

\[
d(fx_2, fw) = \delta(fx_2, W(fx_2, fx_3, \frac{1}{2})) \\
\leq \frac{1}{2} [d(fx_2, fx_2) + d(fx_2, fx_3)] \\
\leq \frac{1}{2} \delta(Fx_2, fx_2) < \frac{1}{2} \delta(fx_0, Fx_0).
\]

(3.3)

It follows from (3.2) and (3.3) that

\[
\delta(Fw, fw) \leq \delta(Fw, W(Fx_1, Fx_2, \frac{1}{2})) \\
\leq \frac{1}{2} [\delta(Fw, Fx_1) + \delta(Fw, Fx_2)] \\
\leq \frac{1}{2} [\phi(ad(fw, fx_1) + (1 - a) \max \{\delta(Fw, fw), \delta(Fx_1, fx_1)\}) \\
+ \phi(ad(fw, fx_2) + (1 - a) \max \{\delta(Fw, fw), \delta(Fx_2, fx_2)\})] \\
< \frac{a + 2}{2} [d(fw, fx_1) + d(fw, fx_2)] \\
+ (1 - a) \max \{\delta(Fx_0, fx_0), \delta(Fw, fw)\} \\
< \frac{a(3 + a)}{4} \delta(Fx_0, fx_0) + (1 - a) \max \{\delta(Fx_0, fx_0), \delta(Fw, fw)\}.
\]

If \( \delta(Fx_0, fx_0) \geq \delta(Fw, fw) \), then

\[
\delta(Fw, fw) < \frac{4 + a^2 - a}{4} \delta(Fx_0, fx_0).
\]

If \( \delta(Fx_0, fx_0) \leq \delta(Fw, fw) \), then

\[
\delta(Fw, fw) < \frac{3 + a}{4} \delta(Fx_0, fx_0).
\]
Take \( \alpha = \max\left\{ \frac{4 + a^2 - a}{4}, \frac{3 + a}{4} \right\} \). It is clear that \( 0 \leq \alpha < 1 \). we obtain that
\[
\delta(Fw, fw) < \alpha \delta(Fx_0, fx_0).
\]
Therefore
\[
\inf\{\delta(Fx_0, fx_0) : x_0 \in K\} \leq \inf\{\delta(Fw, fw) : w = W(fx_2, fx_3, \frac{1}{2})\} < \alpha \inf\{\delta(Fx_0, fx_0) : x_0 \in K\}.
\]
So, \( \inf\{\delta(Fx_0, fx_0) : x_0 \in K\} = 0 \). Hence, we have from Theorem 3.1 (2) that \( F \) and \( f \) have a unique common fixed point \( u \) in \( K \) and \( Fu = \{u\} \).

**Remark 3.1.** In Theorem 3.2, if \( F \) is a single-valued mapping of \( K \) into itself and \( \phi(t) = kt \), for all \( t > 0 \), where \( k \in (0, 1) \), we obtain a generalization of Theorem B for weakly commuting mappings.

**Remark 3.2.** In Theorem 3.2, if \( F \) is a single-valued mapping of \( K \) into itself and \( \phi(t) = kt \), for all \( t > 0 \), where \( k \in (0, 1) \), we obtain a generalization of Theorem 2.1 for compatible mappings of Jungck [16].

Now, we give an example to show the greater generality of Theorem 3.2 over Theorem B.

**Example.** Let \( X = [0, \infty) \) with the Euclidean metric \( d \) and define
\[
f x = x^3 + 3x^2 + 3x, \quad F x = [0, \frac{x^3}{6}],
\]
for all \( x \) in \( X \). Suppose that \( K = [0, 10] \) and \( \phi(t) = \frac{1}{3} t \). For all \( x, y \in X \),
\[
\delta(Fx, Fy) = \max\left\{ \frac{x^3}{6}, \frac{y^3}{6} \right\}
\]
\[
= \frac{1}{3} \frac{1}{2} \max\{x^3, y^3\}
\]
\[
\leq \frac{1}{3} \frac{1}{2} \max\{(x^3 + 3x^2 + 3x), (y^3 + 3y^2 + 3y)\}
\]
\[
= \frac{1}{3} \frac{1}{2} \max\{\delta(fx, Fx), \delta(fy, Fy)\}
\]
\[
\leq \frac{1}{3} \frac{1}{2} d(fx, fy) + (1 - \frac{1}{2}) \max\{\delta(fx, Fx), \delta(fy, Fy)\}
\]
\[
= \phi\left(\frac{1}{2} d(fx, fy) + (1 - \frac{1}{2}) \max\{\delta(fx, Fx), \delta(fy, Fy)\}\right),
\]
i.e., condition (2.3) is satisfied. Also we fined that
\[
f x_n \to 0, \quad F x_n \to \{0\} \quad \text{if} \quad x_n \to 0 \quad \text{and} \quad \delta(F F x_n, f F x_n) \to 0 \quad \text{as} \quad x_n \to 0.
\]
Also, we get \( f F x_n \in B(X) \), i.e., \( f \) and \( F \) are \( \delta \)-compatible and hence they are subcompatible. It is obvious that \( f \) and \( F \) are continuous, \( FK \subseteq fK \) and \( f \) is
surjective. So, all assumptions of Theorem 3.2 satisfy and 0 is the unique common fixed point. Note that the extension of Theorem B to multi-valued mappings is not applicable because \(F\) and \(f\) are not weakly commuting mappings at \(x = 1\) and hence Theorem B is not applicable.

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**References**


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