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A FUNCTIONAL MODEL FOR A FAMILY OF OPERATORS INDUCED BY LAGUERRE OPERATOR

HATAMLEH RA’ED

Abstract. The paper generalizes the instruction, suggested by B. Sz.-Nagy and C. Foias, for operator function induced by the Cauchy problem

\[ T_t : \begin{cases} \quad th''(t) + (1-t)h'(t) + Ah(t) = 0 \\
\quad h(0) = h_0, (th')(0) = h_1 \end{cases} \]

A unitary dilatation for \( T_t \) is constructed in the present paper, then a translational model for the family \( T_t \) is presented using a model construction scheme, suggested by Zolotarev, V., [3]. Finally, we derive a discrete functional model of family \( T_t \) and operator \( A \) applying the Laguerre transform

\[ f(x) \rightarrow \int_0^\infty f(x) P_n(x) e^{-x} dx \]

where \( P_n(x) \) are Laguerre polynomials [6, 7]. We show that the Laguerre transform is a straightening transform which transfers the family \( T_t \) (which is not semigroup) into discrete semigroup \( e^{-itn} \).

Introduction

Functional models for contraction semigroups \( Z_t = \exp(itA) \) and \( T^n, (t \geq 0, n \in \mathbb{Z}^+) \) have been constructed by B. Sz.-Nagy and C. Foias [2] at the beginning of 70-s. The bases of this method is a significant concept of dilatation of contraction semigroup. A spectral realization of the dilatation and subsequent narrowing upon the original space leads to a functional model of the contraction semigroup. As a result an operator \( A(T) \) in this case is realized by operators which carry out multiplication by independent variable in a specific functional space. The basis of the concept is the Fourier transform of space \( L^2 \).

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1. Preliminary information on the functional model in a Fourier representation

1.1. We recall [1] that operator collegation $\Delta,$

$$\Delta = (A, H, \phi, E, \sigma)$$

is a collection of Hilbert spaces $H$ and $E$ and of linear operators $A : H \rightarrow H,$ $\phi : H \rightarrow E,$ $\sigma : E \rightarrow E$ ($\sigma^* = \sigma$) where the collegation condition holds:

$$A - A^* = i\phi^* \sigma \phi.$$  

It is customary to associate with the collegation (1) an open system [1] which is defined by relations

$$\begin{cases} \frac{d}{dt} h(t) + A h(t) = \phi^* \sigma u(t); \\ h(0) = h_0, \ (t \geq 0); \\ v(t) = u(t) - i\phi h(t) \end{cases}$$

where $h(t), u(t), v(t)$ are vector functions from Hilbert spaces $H$ and $E$ respectively. An important role in the further construction of the model representation plays the conservation Law [1].

Theorem 1.1. For the open system (3) associated with the collegation $\Delta$ (1) the conservation Law holds

$$\|h_0\|^2 + \int_0^T \langle \sigma u(\zeta), u(\zeta) \rangle \, d\zeta = \|h(T)\|^2 + \int_0^T \langle \sigma v(\zeta), v(\zeta) \rangle \, d\zeta$$

for any $T, \ 0 \leq T \leq \infty.$

If operator $A$ is selfadjoint then $\phi = 0, \sigma = 0,$ and Cauchy problem (3) in induced by the semigroup

$$Z_t = \exp(itA), \ \text{i.e.} \ h(t) = Z_t h_0$$

and the conservation Law (4) yields $Z_t.$

1.2. Let us consider a contractive semigroup $Z_t = \exp(itA) \ (t \geq 0),$ which has a property $\|Z_t h\| \leq \|h\|$ for all $h \in H.$

A unitary dilatation of contractive semigroup $Z_t$ in $H$ is said to be a unitary semigroup $U_t$ in $\mathcal{H}$ [2] such that the following relation holds:

$$\mathcal{H} \supseteq H; \ P_H U_t|_H = Z_t \quad (t \geq 0)$$

where $P_H$ is an orthoprojector on $H.$ The dilatation $U_t$ in $H$ is said to be minimal if

$$\mathcal{H} = \text{span}\{U_t h; \ t \in \mathbb{R}, \ h \in H\}$$

where span in (6) denotes a closed linear span of the vectors $U_t h$ for any $t \in \mathbb{R}$ and any $h \in H.$

A significant role in the theory of dilatation of contractive semigroup $Z_t$ plays the following Theorem 1.2.
Theorem 1.2. Any contracting semigroup $Z_t$ in $H$ has a unitary dilatation $U_t$ in $H$. Moreover the minimal dilatation $U_t$ is defined up to isomorphism.

We present a construction of the dilatation $U_t$ according to the paper [3]. A contractibility of the semigroup $Z_t$ means [2, 3] that $A$ is dissipative, i.e. $-i(A - A^*) \geq 0$. Consequently including $A$ into the collegation $\Delta$ (1) we can assume that $\sigma = I$. Therefore the conservation law (4) has the form

$$\left\| h_0 \right\|^2 + \int_0^T \left\| u(\zeta) \right\|^2 d\zeta = \left\| h(T) \right\|^2 + \int_0^T \left\| v(\zeta) \right\| d\zeta$$

We defined [3] a dilatation space $H$, which forms vector-functions $f(\zeta) = (u_+(\zeta), h, u_-(\zeta))$ so that $u_\pm(\zeta) \in E$ and Supp $u_\pm(\zeta) \in \mathbb{R}^\pm$ for a finite norm

$$\left\| f \right\|^2 = \int_{-\infty}^0 \left\| u_+(\zeta) \right\|^2 d\zeta + \left\| h \right\|^2 + \int_0^{\infty} \left\| u_-(\zeta) \right\|^2 d\zeta < \infty.$$ 

We define a dilatation $U_t$ in $H$ by the formula

$$(U_t f)(\zeta) = (u_+(t, \zeta), h_t, u_-(t, \zeta))$$

where $u_-(t, \zeta) = P_{\mathbb{R}^+} u_-(\zeta + t)$; $h_t = y_t(0)$, and $y_t(\zeta)$ is a solution of the Cauchy problem

$$\begin{cases} i \frac{d}{d\zeta} y_t(\zeta) + Ay_t(\zeta) = \phi^* u_-(\zeta + t); \\ y_t(-t) = 0, \quad \zeta \in (-t, 0); \end{cases}$$

and at last $u_+(t, \zeta) = u_+(t + \zeta) + P_{(-t, 0)} \{ u_-(\zeta + t) - i\phi y_t(\zeta) \}$ where $P_{\mathbb{R}^+}$ and $P_{(-t, 0)}$ are operators of narrowing (projection operators at set $\mathbb{R}^+$ and $(-t, 0)$ respectively), $t \geq 0$.

It is not difficult to show that unitary of $U_t$ (9) in $H$ is a consequence of the conservation law (1). By the dilatation construction $U_t$ one can see that the space $H$ has the form

$$H = D_+ + H \oplus D_-$$

where the subspace $D_+$ is found by vector-function of the form $(u_+(\zeta), 0, 0) \in H$ and the subspace $D_-$ is formed by vector-function $(0, 0, u_-(\zeta))$ from $H$, respectively.

The subspaces $D_\pm$ have the following properties:

$$U_t D_+ \subseteq D_+ \quad (t \geq 0),$$

$$U_t D_- \subseteq D_- \quad (t \leq 0).$$

Thus $D_+$ is outgoing subspace and $D_-$ is incomming subspace in the sense of P.D. Lax and R.S. Phillips [4]. In accordance with the paper [3], we define a free unitary group $V_t$ in the space $L^2_{\mathbb{R}}(E)$, which will act as

$$(V_t g)(\zeta) = g(\zeta + t)$$
and vector-function \( g(\zeta) \in E, \zeta \in \mathbb{R} \) is such that
\[
\int_{-\infty}^{\infty} \|g(\zeta)\|^2 d\zeta < \infty.
\]
It is evidently that \( D_\pm \) after identification belongs to \( L^2_{\mathbb{R}}(E) \) also.

Wave operators \( W_\pm \) play a significant role in the scattering theory. They are defined \([3, 4]\) as
\[
W_\pm = s - \lim_{t \to \mp \infty} U_t P_{D_\pm} V_{-t}
\]
where \( P_{D_\pm} \) are orthoprojectors on subspaces \( D_\pm \). The following theorem holds \([3]\).

**Theorem 1.3.** The wave operators \( W_\pm \) exist as strong limits (13) are isometries from \( L^2_{\mathbb{R}}(E) \) to \( \mathcal{H} \), and the relations
\[
W_\pm V_t = U_t W_\pm, \quad (\forall t), \quad W_\pm P_{D_\pm} = P_{D_\pm}
\]
are valid.

The scattering operator \( S \) is defined by the wave operator \( W_\pm \) in a conventional way \([3, 4]\):
\[
S = W_+^* W_-
\]
From Theorem 1.3 there follows a proposition.

**Theorem 1.4.** The operator \( S \) (15) is a contraction, i.e. \( \|S\| \leq 1 \) and has the properties:
\[
SV_t = V_t S; \quad SL^2_{\mathbb{R}+} \subseteq L^2_{\mathbb{R}+}(E);
\]
\[
SL^2_{\mathbb{R}}(E) = L^2_{\mathbb{R}}(E)
\]
1.3. We recall that the collegation \( \Delta \) (1) is simple \([1–3]\) if \( H = \text{span}\{A^n \phi^* g; n \in \mathbb{Z}_+, g \in E\} \}. Let us define the following subspaces in \( \mathcal{H} \),
\[
\mathcal{R}_\pm = \overline{W_\pm L^2_{\mathbb{R}}(E)}.
\]
The following theorem gives a sufficient condition for the completeness of the wave operators \( W_\pm \), \([3]\).

**Theorem 1.5.** If the collegation \( \Delta \) is simple then the relation \( \mathcal{H} = \text{span}\{f_+ + f_-; f_\pm \in \mathcal{R}_\pm\} \) holds.

Now we construct a translational model \([3]\). Let \( f_k(\zeta) \in L^2_{\mathbb{R}}(E), (k = 1, 2) \). We define a mapping
\[
\begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix} \rightarrow \Psi_{p}(\zeta) = W_- f_1(\zeta) + W_+ f_2(\zeta) \in \mathcal{H}.
\]
Then using isometry of $W_\pm$ and the form of operator $S$ (15) it is not difficult to show that

\[ \|\Psi_p(\zeta)\|^2 = \int_{-\infty}^{\infty} < \begin{bmatrix} I & S^* \\ S & I \end{bmatrix} \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix}, \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix} > d\zeta, \]

Using Theorem 1.5 we may assert, that space $H$ is isomorphic to the space $L^2\left(\begin{bmatrix} 1 & S^* \\ S & 1 \end{bmatrix}\right)$ which is formed by vector-functions $f(\zeta) = \begin{pmatrix} f_1(\zeta) \\ f_2(\zeta) \end{pmatrix}$ for which the norm (17) is finite. By virtue of conditions (14) the dilatation $U_t$ on $\Psi_p$ will act as a shift. Therefore if $f(\zeta) \in L^2\left(\begin{bmatrix} 1 & S^* \\ S & 1 \end{bmatrix}\right)$ then the dilatation $U_t$ is transformed into

\[ \hat{U}_t f(\zeta) = f(\zeta + t). \]

Applying again (14), one can easily deduce that the spaces $D_\pm$ are realized now in the form

\[ \hat{D}_- = \begin{pmatrix} L^2_{\mathbb{R}^+}(E) \\ 0 \end{pmatrix}, \quad \hat{D}_+ = \begin{pmatrix} 0 \\ L^2_{\mathbb{R}^-}(E) \end{pmatrix}. \]

Thus the initial space $H$ acquires such model form

\[ \hat{H}_p = L^2\left(\begin{bmatrix} 1 & S^* \\ S & 1 \end{bmatrix}\right) \ominus \begin{pmatrix} L^2_{\mathbb{R}^+}(E) \\ L^2_{\mathbb{R}^-}(E) \end{pmatrix} = f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in L^2\left(\begin{bmatrix} 1 & S^* \\ S & 1 \end{bmatrix}\right); \quad f_1 + S^* f_2 \in L^2_{\mathbb{R}^-}(E) \quad \text{and} \quad S f_1 + f_2 \in L^2_{\mathbb{R}^+}(E) \]

and in the virtue of the dilatation the action of semigroup $Z_t$ is transformed to the shift semigroup

\[ \hat{Z} f(\zeta) = P_{\hat{H}_p} f(\zeta + t) \]

where $f(\zeta) \in \hat{H}_p$ (20). Thus the following theorem is proved.

**Theorem 1.6.** A minimal unitary dilatation $U_t$ in $\mathcal{H}$ of the contraction semigroup $Z_t = \exp(itA)$ in $H$, where $A$ is dissipative operator of a simple collegation $\Delta$ is unitary equivalent to a translation group $\hat{U}_t$ (18) in the space $L^2\left(\begin{bmatrix} 1 & S^* \\ S & 1 \end{bmatrix}\right)$, and the contraction semigroup $Z_t$ is unitary equivalent to the shift semigroup $\hat{Z}_t$ (21) in the space $\hat{H}_p$, respectively.

The Fourier transform by formula

\[ \tilde{f}(\lambda) = \int_{-\infty}^{\infty} f(\zeta) e^{-i\lambda \zeta} d\zeta \]
in the virtue of Plancherel theorem [2, 3] is a unitary operator in $L^2_{\mathbb{R}}(E)$. By the virtue of Wiener-Paley theorem
\[ \tilde{L}^2_{\mathbb{R}^+}(E) = H^2(E); \quad \tilde{L}^2_{\mathbb{R}^-}(E) = H^2_+(E) \]
where $H^2_+(E)$ are Hardy spaces of $E$-value function from $L^2_{\mathbb{R}}(E)$ which are holomorphically continued into lower (upper) half-plane. Let us apply the Fourier transform (22) to translational model (18) – (21) and take advantage of the following Theorem 1.7

**Theorem 1.7.** The Fourier transform of the scattering operator $S$ (15) transfers the operator $S$ into operator performing multiplication by characteristic function
\begin{equation}
S_\Delta(\lambda) = I - \phi(A - \lambda I)^{-1} \phi^*, \quad \text{i.e.}
\end{equation}
\begin{equation}
(\hat{S}f)(\lambda) = S_\Delta(\lambda)\tilde{f}(\lambda).
\end{equation}
As it is known $\tilde{f}(\lambda + t) = e^{i\lambda t} \tilde{f}(\lambda)$, therefore we derive such functional model.

**Theorem 1.8.** A minimal unitary dilatation $U_t$ in $H$ of the contraction semigroup $Z_t = \exp(itA)$ in $H$, where $A$ is dissipative operator of a simple collegation $\Delta$ is unitary equivalent to the group
\begin{equation}
\tilde{U}_t f(\lambda) = e^{i\lambda t} f(\lambda)
\end{equation}
where $f(\lambda) \in L^2 \left( \begin{array}{c} I \\ S_\Delta(\lambda) \\ S^*_\Delta(\lambda) \\ I \end{array} \right)$ and contraction semigroup $Z_t$ is unitary equivalent to semigroup $\tilde{Z}_t f(\lambda) = P_{\tilde{H}_p} e^{i\lambda t} f(\lambda)$, where $f(\lambda)$ belongs to the space
\[ \tilde{H}_p = \left\{ f = \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right)(\lambda) \in \left( \begin{array}{c} I \\ S_\Delta(\lambda) \\ S^*_\Delta(\lambda) \\ I \end{array} \right) ; \quad f_1 + S^*_\Delta(\lambda)f_2 \in H^2_+(E), \quad S_\Delta(\lambda)f_1 + f_2 \in H^2_-(E) \right\} \]
Here the main operator $\tilde{\Lambda}$ in $\tilde{H}_p$ act as multiplication operator by independent variable
\begin{equation}
\tilde{\Lambda} f(\lambda) = P_{\tilde{H}_p} \lambda f(\lambda), \quad f(\lambda) \in \tilde{H}_p.
\end{equation}
In the next section we will generalize this construction on the case of the Laguerre transform.

2. A functional model for the Laguerre representation

2.1. Let us consider a differential operator
\begin{equation}
\ell = t \frac{d^2}{dt^2} + (1 - t) \frac{d}{dt}
\end{equation}
in what follows called the Laguerre operator; it acts on functions form $C^2 = (\mathbb{R}_+)$. We denote by $L^2_{\mathbb{R}_+}(e^{-t} dt)$ the following space:
\begin{equation}
L^2_{\mathbb{R}_+}(e^{it} dt) = \{ f(t), \ t \in \mathbb{R}_+; \int_0^\infty |f(t)|^2 e^{-t} dt < \infty \}.
\end{equation}
Proposition 2.1. An operator \( \ell \) is symmetric in the space \( L^2_{\mathbb{R}^+} (e^{-t} dt) \) under the self-adjoint boundary conditions, i.e. \( \langle \ell x, y \rangle = \langle y, \ell x \rangle \) for all \( x, y \in C^2(\mathbb{R}^+) \) such that \( tx(t)|_{t=0} = 0, ty(t)|_{t=0} = 0 \) and \( ty'(t)|_{t=0} < \infty, tx'(t)|_{t=0} < \infty. \)

Proof. We calculate
\[
\langle \ell x, y \rangle - \langle x, \ell y \rangle = \int_0^\infty \{ (tx'' + (1 - t)x')\hat{y} - x(\hat{t}y'' + (1 - t)y') \} e^{-t} dt
\]
\[
= \int_0^\infty \{ te^{-t}(x'\hat{y} - \hat{y}'x) \}' dt = \{ te^{-t}(x'\hat{y} - \hat{y}'x) \}|_0^\infty = 0
\]
by virtue of the boundary conditions.

Let us consider now an open system of special form, generated by the Laguerre operator (27) and corresponding to the collegation \( \Delta (1) \):
\[
\begin{align*}
\ell h(t) + Ah(t) &= \phi^* \sigma u(t); \\
h(0) &= h_0(\theta h')(0) = h_1; \\
v(t) &= u(t) - i\phi h(t).
\end{align*}
\]
(29)

The following assertion is valid, similar to Theorem (1.1).

Theorem 2.1. For the open system (29) associated with collegation \( \Delta \) the law of conservation of energy is valid, i.e.
\[
\begin{align*}
\int_0^T \langle \sigma u(\zeta), u(\zeta) \rangle e^{-\zeta} d\zeta + \langle I\hat{h}_0, \hat{h}_0 \rangle &= \int_0^T \langle \sigma v(\zeta), v(\zeta) \rangle e^{-\zeta} d\zeta + \langle I\hat{h}_T, \hat{h}_T \rangle
\end{align*}
\]
(30)

where \( I = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \) and \( h_0 = \begin{pmatrix} h_0 \\ h_T \end{pmatrix} \), \( h_T = \left( \begin{array}{c} h(T) \\ e^{-T} h'(T) \end{array} \right) \) for any finite \( T > 0. \)

Proof. We calculate
\[
\langle \ell h, h \rangle - \langle h, \ell h \rangle = \langle \phi^* \sigma u - Ah, h \rangle - \langle h, \psi^* \sigma u - Ah \rangle
\]
\[
= \langle \sigma u, \frac{u - v}{i} \rangle - \langle \frac{u - v}{i}, \sigma u \rangle - \langle (A - A^*)h, h \rangle
\]
\[
= i\langle \sigma u, u - v \rangle + i\langle u - v, \sigma u \rangle - i\langle \phi^* \sigma h, h \rangle
\]
\[
= i\langle \sigma u, u - v \rangle + i\langle u - v, \sigma u \rangle - i\langle \sigma(u - v), u - v \rangle
\]
\[
= i\langle \sigma u, u \rangle - i\langle \sigma v, v \rangle.
\]

Now we integrate the derived equality:
\[
\begin{align*}
\int_0^T \langle \sigma v, v \rangle e^{-t} dt - \int_0^T \langle \sigma u, u \rangle e^{-t} dt
\end{align*}
\]
\[
= i\int_0^T \left( \langle \ell h, h \rangle - \langle h, \ell h \rangle \right) e^{-t} dt
\]
\[
= i\left( e^{-t} \left( \langle h'', h \rangle - \langle h', h' \rangle \right) \right)|_0^T
\]
\[
= \langle I\hat{h}_0, \hat{h}_0 \rangle - \langle I\hat{h}_T, \hat{h}_T \rangle
\]
which proves our assertion. □

2.2. Let us make use of the energy conservation law (30) to construct a dilatation for operator \( T_t \) generated by the Cauchy problem

\[
\begin{cases}
\ell h(t) + Ah(t) = 0; \\
h(0) = h_0; \ (th')(0) = h_1;
\end{cases}
\]

where \( T_t(h_0, h_1) = (h(t), th')(t) \). We will call an unitary operator-function \( U_t \) in \( H \) a dilatation of family \( T_t \) in \( H \), if \( H \supseteq H, T_t = P_H U_t|_H \).

Here we do not suppose that \( T_t \) and \( U_t \) is semigroup. Moreover, the unitary property of \( U_t \) may hold not necessarily in Hilbert metric but in indefinite one. The following analog of Theorem 1.2 is valid.

**Theorem 2.2.** The operator-function \( T_t \) generated by the Cauchy problem (31) with dissipative operator \( A \) of collegation \( \Delta (1) \) (i.e. \( \sigma = I \)) possesses the unitary (in indefinite metric) dilatation \( U_t \), where the minimal dilatation is determined up to isomorphism.

**Proof.** To prove the theorem we bring a construction of dilatation \( U_t \) by analog with (8), (9).

Let us consider a Hilbert space

\[
\mathcal{H} = \{ f = (u(\zeta), \hat{h}, v(\zeta)); \ u(\zeta), v(\zeta) \in E, \ \text{supp} \ v \in \mathbb{R}_-, \ \text{supp} \ u \in \mathbb{R}_+, \ \hat{h} = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix}, h_k \in H; \ ||f||^2 = \int_{-\infty}^{0} ||v(\zeta)||^2 e^{-\zeta} d\zeta + ||h||^2 + \int_{0}^{\infty} ||u(\zeta)||^2 e^{-\zeta} d\zeta < \infty \}.
\]

We set indefinite metric \( \mathcal{H} \)

\[
\langle f \rangle_I^2 = \int_{-\infty}^{0} ||v(\zeta)||^2 e^{-\zeta} d\zeta + \langle I\hat{h}, \hat{h} \rangle + \int_{0}^{\infty} ||u(\zeta)||^2 e^{-\zeta} d\zeta
\]

where \( I \) has the form indicated in Theorem 2.1.

We construct the dilatation \( U_t \) in \( \mathcal{H} \),

\[
U_t f = f_t(u(t, \zeta), \hat{h}_t, v(t, \zeta)).
\]

Let us consider further the Cauchy problem

\[
\begin{cases}
\left( i \frac{\partial}{\partial t} + \ell_\zeta \right) \hat{u}(t, \zeta) = 0; \\
\hat{u}(0, \zeta) = u(\zeta); \ \zeta \in \mathbb{R}_+;
\end{cases}
\]

where \( \ell_\zeta \) is operator \( \ell \) (27) with respect to \( \zeta \).
Solution of the problem is easily obtained. In fact, let

\[ \hat{u}(t, \zeta) = \sum_{n \in \mathbb{Z}^+} e^{-itn} C_n g_n(\zeta) \]

where \( g_n(\zeta) \) are the Laguerre polynomials [5] which are the solutions of equation \( \ell_\zeta g_n(\zeta) + ng_n(\zeta) = 0 \) and have the form

\[ g_n(\zeta) = \frac{1}{n!} e^\zeta \frac{d^n}{d\zeta^n} (\zeta e^{-\zeta}) \]

and make a complete system of orthogonal polynomials in \( L^2_{\mathbb{R}^+}(e^{-\zeta} d\zeta) \). The coefficients \( C_n \) are obtained from the initial condition \( \sum C_n g_n(\zeta) = u(\zeta) \).

Therefore \( \hat{u}(t, \zeta) \) possesses the property \( \text{supp} \hat{u}(t, \zeta) = \text{supp} \hat{u}(\zeta) \subseteq \mathbb{R}^+ \). Now we determine \( u(t, \zeta) \) in (34) by the formula

\[ u(t, \zeta) = P_{\mathbb{R}^+} \hat{u}(t, \zeta + t)e^{-\frac{\zeta}{2}}. \]

To set \( \hat{h}_t \) (34), we consider the following Cauchy problem

\[
\begin{cases}
\ell_\zeta y(\zeta) + Ay(\zeta) = \phi^* \hat{u}(t, \zeta + t)e^{-\frac{\zeta}{2}}; \quad \zeta \in (-t, 0); \\
y(-t) = h_0; \\
(-t)e^t y(-t) = h_1;
\end{cases}
\]

and put \( \hat{h}_t = \left( \begin{array}{c} y(0) \\ (ty')(0) \end{array} \right) \).

Finally, to set \( v(t, \zeta) \) (34) we consider the similar equation

\[
\begin{cases}
(i \frac{\partial}{\partial t} + \ell_\zeta) \hat{v}(t, \zeta) = 0; \\
\hat{v}(0, \zeta) = v(\zeta); \quad \zeta \in \mathbb{R}^-;
\end{cases}
\]

and put \( v(t, \zeta) = e^{-\frac{\zeta}{2}} \hat{v}(t, \zeta + t) + P_{\mathbb{R}^-} \{ \hat{u}(t, \zeta + t)e^{-\frac{\zeta}{2}} - i\phi y(\zeta) \} \). We show that \( U_t \) (34) has property of isometry in the metric (33). To this end we calculate,

\[
\langle f \rangle_I^2 = \int_{-\infty}^{0} \|v(t, \zeta)\|^2 e^{-\zeta} d\zeta + \langle \hat{I}\hat{h}_t, \hat{h}_t \rangle + \int_{0}^{\infty} \|u(t, \zeta)\|^2 e^{-\zeta} d\zeta
\]

\[
= \int_{-\infty}^{-t} \|\hat{v}(t, \zeta + t)\|^2 e^{-\zeta-t} d\zeta + \int_{-t}^{0} \|\hat{u}(t, \zeta + t)e^{-\frac{\zeta}{2}} - i\phi y(\zeta)\|^2 e^\zeta d\zeta
\]

\[
+ \langle \hat{I}\hat{h}_t, \hat{h}_t \rangle + \int_{0}^{\infty} \|u(t, \zeta + t)\|^2 e^{-\zeta-t} d\zeta
\]

\[
= \int_{-\infty}^{-t} \|\hat{v}(t, \zeta + t)\|^2 e^{-\zeta-t} d\zeta + \langle \hat{I}\hat{h}_0, \hat{h}_0 \rangle + \int_{-t}^{0} \|\hat{u}(t, \zeta + t)\|^2 e^{-\zeta-t} d\zeta
\]

\[
= \int_{-\infty}^{0} \|\hat{v}(t, \zeta)\|^2 e^{-\zeta} d\zeta + \langle \hat{I}\hat{h}_0, \hat{h}_0 \rangle + \int_{0}^{\infty} \|\hat{u}(t, \zeta)\|^2 e^{-\zeta} d\zeta
\]

\[
= \langle f \rangle_I^2
\]
In this calculation we have made use of the conservation law (30) and of the fact that norms of solutions of Cauchy problems \( \hat{u}(t, \zeta), \hat{v}(t, \zeta) \) (35) and (38) coincide with norms of initial data \( u(\zeta) \) and \( v(\zeta) \) in the spaces \( L^2_{\mathbb{R}_+}(e^{-t} dt) \) and \( L^2_{\mathbb{R}_-}(e^{-t} dt) \) by virtue of selfadjointness of operators \( \ell_{\zeta} \) in the spaces.

In order to prove that \( U_t \) has the property of being unitary, it is necessary to ascertain that from \( U_t^* f = 0 \) implies \( f = 0 \). It is easy to show that \( U_t^* \) will act by the formula

\[
U_t^* f = (u(t, \zeta), \hat{h}_t, v(t, \zeta)).
\]

Here \( v(t, \zeta) = P_{\mathbb{R}_-} \hat{v}(t, \zeta - t)e^{\frac{t}{2}} \) where \( \hat{v}(t, \zeta) \) is a solution of problem (38).

In order to obtain \( \hat{h}_t \), it is necessary to consider dual to (37) problem

\[
\begin{cases}
  \ell_{\zeta} y(\zeta) + A^* y(\zeta) = \phi^* \hat{v}(\zeta, \zeta - t)e^{\frac{t}{2}}; \\
  y(t) = h_0; \\
  e^{-t} y'(t) = h_1;
\end{cases}
\]

and put \( \hat{h}_t = \begin{pmatrix} y(0) \\ ty'(0) \end{pmatrix} \). Finally,

\[
u(t, \zeta) = \hat{u}(t, \zeta - t)e^{\frac{t}{2}} + P_{\mathbb{R}_+} \{ \hat{v}(t, \zeta - t)e^{\frac{t}{2}} + i\phi y(\zeta) \},
\]

where \( \hat{u}(t, \zeta) \) is the solution of Cauchy problem (35).

Thus let \( U_t^* f = 0 \), then \( \hat{u}(t, \zeta) = 0 \) and so \( \hat{u}(t, \zeta) = 0 \) and \( \hat{v}(t, \zeta - t)e^{\frac{t}{2}} + i\phi y(\zeta) = 0 \) therefore \( u(\zeta) \equiv 0 \). Now, by substituting \( \hat{v}(t, \zeta - t) = -i\phi y(\zeta)e^{-\frac{t}{2}} \) in (40) we obtain a homogeneous equation

\[
\ell_{\zeta} y + A^* y + i\phi^* \phi y = 0
\]

with zero condition in the origin \( \hat{h}_t = 0 \). By virtue of uniqueness of Cauchy problem solution, this yields that \( y(\zeta) \equiv 0 \), therefore \( \hat{v}(t, \zeta - t) = 0 \) on interval \((0, t)\). Accounting that \( \hat{v}(t, \zeta - t) = 0 \) with \((-\infty, 0)\), finally we conclude that \( v(\zeta) = 0 \). Thus \( f = 0 \). This proves the property of being unitary for \( U_t \) (34) and completes the proof of the theorem.

2.3. Let us pass to constructing wave operators. To this end we define a “free” group by analogy with (38)

\[
V_t h(\zeta) = g(t, \zeta),
\]

where \( g(t, \zeta) \) is a solution of Cauchy problem

\[
\begin{cases}
  (i \frac{\partial}{\partial t} + \ell_{\zeta}) g(t, \zeta) = 0; \\
  g(0, \zeta) = g(\zeta) \in L^2_{\mathbb{R}}(e^{-\zeta} d\zeta).
\end{cases}
\]

It is evident that \( V_t \) (41) is unitary. Now we define the operators

\[
W_- = s - \lim_{t \to +\infty} U_t P_{\mathbb{R}_+} V_{-t},
\]

\[
W_+ = s - \lim_{t \to -\infty} U_t^* P_{\mathbb{R}_-} V_{-t}^*.
\]

By analogy with Theorem 1.3 we have
Theorem 2.3. The wave operators $W_{\pm}$ exist as strong limits (43), are isometries from $L^2_{\mathbb{R}}(e^{-\xi}d\xi)$ to $\mathcal{H}$, and the following relations are valid:

\begin{align}
U_t W_- &= W_- V_t, \quad U_t^* W_+ = W_+ V_t^*, \quad (t \geq 0) \\
W_\pm P_{\mathbb{R}_\mp} &= P_{\mathbb{R}_\mp}.
\end{align}

\[\text{(44)}\]

**Proof.** We prove the assertion of the theorem for $W_-$ (for $W_+$ the proof is similar). The main matter of the theorem consists of existence proof of $W_-$ since the relation (44) is proved by analogy with arguments given in Section 1; see [2, 3]. Let

$$f_t = U_t P_{\mathbb{R}_+} V_{-t} g = (v(t, \zeta), h_t, u(t, \zeta))$$

then $u(t, \zeta) = P_{\mathbb{R}_+} g(\zeta)$. We consider the Cauchy problem

\[\begin{align}
\ell_{\zeta} y(\zeta) + A y(\zeta) &= \phi^* g(\zeta) ; \\
y(-t) &= 0; \quad y'(-t) = 0, \quad \zeta \in (-t, 0).
\end{align}\]

Then $\hat{h}_t = \left( \begin{array}{c} y(0) \\ (ty')(0) \end{array} \right)$.

We denote by $K(\zeta, \eta)$ a Cauchy function of the problem (45) (i.e. $K(\zeta, \zeta) = 0$, $K'(\zeta, \zeta) = I$), then a solution $y(\zeta)$ of (45) has the form

$$y_t(\zeta) = \int_{-t}^{\zeta} K(\zeta, \eta) \phi^* g(\eta) d\eta.$$

Therefore $V(t, \zeta)$ has the form

$$V(t, \zeta) = P_{(-t, 0)} \{ g(\zeta) - i \phi y(\zeta) \}.$$

Thus,

$$f_t = \left( P_{(-t, 0)} \{ g(\zeta) - i \phi \int_{-t}^{0} K(\zeta, \eta) \phi^* g(\eta) d\eta \}, \left( \int_{-t}^{0} K(0, \eta) \phi^* g(\eta) d\eta, \int_{-t}^{0} K'(0, \eta) \phi^* g(\eta) d\eta \right), P_{\mathbb{R}_+} g(\zeta) \right).$$

We show that $f_t$ is a Cauchy sequence, i.e $\|f_{t+\Delta} - f_t\|^2 \to 0$ as $t \to \infty$. Since

$$\|f_{t+\Delta} - f_t\|^2 = \int_{-\infty}^{0} \|v_t(t + \Delta, \zeta) - v(t, \zeta)\|^2 e^{-\zeta} d\zeta + \|\hat{h}_{t+\Delta} - \hat{h}_t\|^2. \quad \text{(46)}$$

It is sufficient to show that each summand approaches to zero as $t \to \infty$. We show that $\|\hat{h}_{t+\Delta} - \hat{h}\| \to 0$ when $t \to \infty$ and we will prove this property component by component. It is obvious that

$$\|\hat{h}_{t+\Delta} - \hat{h}\|^2 \leq \int_{-t-\Delta}^{-t} \|K(0, \eta)\|^2 e^\eta d\eta \cdot \int_{-t-\Delta}^{-t} e^{-\eta} \|\phi^*\|^2 \|g(\eta)\|^2 d\eta$$

since the relation

$$\left( \int_{-t}^{0} K(0, \eta) \phi^* g(\eta) d\eta, \int_{-t}^{0} K'(0, \eta) \phi^* g(\eta) d\eta \right),$$

is isometries.
and since the function $K(0, \eta)e^\eta$ is bounded (see [6, 7]), we obtain that

$$\|h_{t+\Delta} - h_t\|^2 \leq \Delta C\|\phi^*\|^2 \int_{-t-\Delta}^{-t} \|g(\eta)\|^2 e^{-\eta} d\eta \to 0 \quad \text{as} \quad t \to \infty$$

since $g(\eta) \in L^2_\mathbb{R}(e^{-\eta} d\eta)$.

The convergence of second components $\hat{h}_{t+\Delta} - \hat{h}_t$ to zero is proved in a similar way. We show that the first summand in (46) approaches to zero too.

In fact,

$$A = \int_{-\infty}^0 \|P_{(-t-\Delta, -t)}g(\zeta) - iP_{(-t-\Delta, 0)}\phi \int_{-t-\Delta}^{\zeta} K(\zeta, \eta)\phi^* g(\eta) d\eta$$

$$+ i \int_{-t}^{\zeta} \phi K(\zeta, \eta)\phi^* g(\eta) d\eta\|e^{-\zeta} d\zeta$$

$$= \int_{-t-\Delta}^{-t} \|g(\zeta)\|^2 e^{-\zeta} d\zeta + \int_{-\infty}^0 \|P_{(-t-\Delta, 0)}\phi y_{t+\Delta}(\zeta) - P_{(-t, 0)}\phi y_t(\zeta)\|^2 e^{-\zeta} d\zeta$$

$$+ 2\text{Im} \int_{-t-\Delta}^{-t} \langle g(\zeta), P_{(-t-\Delta, 0)}\phi y(\zeta) - P_{(-t, 0)}\phi y(\zeta)\rangle e^{-\zeta} d\zeta$$

It is obvious that the first and third summands in the given sum approaches to zero as $t \to \infty$ because $g(\zeta) \in L^2_\mathbb{R}(e^{-\zeta} d\zeta)$. We evaluate the second summand:

$$B = \int_{-\infty}^0 \|P_{(-t-\Delta, 0)}\phi y_{t+\Delta}(\zeta) - P_{(-t, 0)}\phi y_t(\zeta)\|^2 e^{-\zeta} d\zeta$$

$$= \int_{-\infty}^0 \langle \phi \Delta y, \phi \Delta y \rangle e^{-\zeta} d\zeta,$$

where

$$\Delta y = P_{(-t-\Delta, 0)}y_{t+\Delta}(\zeta) - P_{(-t, 0)}y_t(\zeta).$$

Then

$$A = \int_{-\infty}^0 \langle \phi^* \phi \Delta y, \Delta y \rangle e^{-\zeta} d\zeta = \int_{-\infty}^0 \left\langle \frac{A - A^*}{i} \Delta y, \Delta y \right\rangle e^{-\zeta} d\zeta$$

$$= 2\text{Im} \int_{-\infty}^0 \langle \phi^* g - \ell \Delta y, \Delta y \rangle e^{-\zeta} d\zeta$$

$$= 2\text{Im} \int_{-\infty}^0 \langle \phi^* g, \Delta y \rangle e^{-\zeta} d\zeta + 2\text{Im} \int_{-\infty}^0 \langle \ell \Delta y, \Delta y \rangle e^{-\zeta} d\zeta$$

the first summand approaches to zero again on account of $g(\zeta) \in L^2_\mathbb{R}(e^{-\zeta} d\zeta)$, and the second one yields after integration by parts

$$\|\zeta e^{-\zeta} \Delta y\| |_{\zeta=0} \to 0 \quad (t \to \infty)$$

since $\Delta \hat{h}_t \to 0$. The theorem is proved. \qed

As before, we define the operator $S$ by the formula (15). Then the following theorem holds.
Theorem 2.4. The operator \( S \) (15) is a contraction from \( L^2_\mathbb{R}(e^{-\zeta}d\zeta) \) to \( L^2_\mathbb{R}(e^{-\zeta}d\zeta) \) and possesses the following properties:

\[
SV_t = V_t S; \quad SL^2_\mathbb{R}(e^{-\zeta}d\zeta) \subset L^2_\mathbb{R}(e^{-\zeta}d\zeta);
\]

\[
SL^2_\mathbb{R}(e^{-\zeta}d\zeta) = L^2_\mathbb{R}(e^{-\zeta}d\zeta).
\]

2.4. Further we suppose that the collegation \( \Delta \) (1) is simple and as in subsection 1.3 we set a mapping

\[
\Psi_p(\zeta = W_- f_1(\zeta)) + W_+ f_2(\zeta)
\]

from \( L^2_\mathbb{R}(e^{-\zeta}d\zeta) + L^2_\mathbb{R}(e^{-\zeta}d\zeta) \) to \( \mathcal{H} \). It is obvious that

\[
\Psi_p(\zeta) \in L^2\left(\begin{pmatrix} I & S^* \\ S & I \end{pmatrix}, e^{-\zeta}d\zeta\right)
\]

Action of dilatation in this space again reduces to a translation

(47)

\[
\hat{U}_t f(\zeta) = f(\zeta + t),
\]

since

\[
U_t \Psi_p(\zeta) = W_- f_1(\zeta + t) + U_t W_+ f_2(\zeta)
\]

\[
= W_- f_1(\zeta + t) + U_t W_+ V_t^* V_t f_2(\zeta)
\]

\[
= W_- f_1(\zeta + t) + U_t U_t^* W_+ V_t f_2(\zeta) = \Psi_p(\zeta + t).
\]

As earlier, it is obvious that

\[
D_- = \begin{pmatrix} L^2_\mathbb{R}_+(e^{-\zeta}d\zeta) \\ 0 \end{pmatrix}, \quad D_+ = \begin{pmatrix} 0 \\ L^2_\mathbb{R}_-(e^{-\zeta}d\zeta) \end{pmatrix}
\]

and the model space \( H_p \) has the form

(48)

\[
H_p = L^2\left(\begin{pmatrix} 1 & S^* \\ S & 1 \end{pmatrix} e^{-\zeta}d\zeta\right) \ominus L^2_\mathbb{R}_+(e^{-\zeta}d\zeta) \ominus L^2_\mathbb{R}_-(e^{-\zeta}d\zeta)
\]

and in addition \( T_t \) passes to shift semigroup

(49)

\[
\hat{T}_t f(\zeta) = f(\zeta + t).
\]

Now we consider a Laguerre transform

(50)

\[
L_n = \int_0^\infty e^{-x} P_n(x) f(x) dx
\]

where \( P_n(x) = \frac{1}{n!} e^{-x} \frac{d^n}{dx^n} (xe^{-x}) \) are a Laguerre polynomials, and \( f(x) \in L^2_\mathbb{R}_+(e^{-x}dx) \). The transform (50) ascertains isomorphism between \( L^2_\mathbb{R}_+(e^{-x}dx) \) and \( \ell^2 \).

We extend the Laguerre transform (50) on \( \mathbb{R}_- \) in a symmetric way. Then an image of this map yields a space \( \ell^2_+ \). Let \( \ell^2_2 = \ell^2_+ + \ell^2_- \) is a space of square summable two-sided sequences. Just as for the case of Fourier transform (see Theorem 1.7 in Section 1) a theorem the proof of which repeats the reasonings brought out in [3] holds.
Theorem 2.5. The Laguerre transform of scattering operator $S$ transfers the operator $S$ into an operator of multiplication by a characteristic function $S_\Delta(n) = I - i\phi(A - nI)^{-1}\phi^*$, $n \in \mathbb{Z}$, i.e.

\begin{equation}
L_n(Sg) = S_\Delta(n)g_n
\end{equation}

where $g_n = L_n(g)$.

After realizing the Laguerre transform, the space $L^2 \left( \left( \begin{array}{cc} I & S^* \\ S & I \end{array} \right) e^{-\zeta} d\zeta \right)$ passes into the space $\ell^2_\mathbb{Z} \left( \begin{array}{cc} I \\ S_\Delta(n) \\ S^*_\Delta(n) \\ I \end{array} \right)$ and dilatation $\hat{U}_t$ (47) is converted into

\begin{equation}
\hat{U}_t(n)f_n = e^{-itn}f_n.
\end{equation}

Supspaces $D_\pm$ will have the form

\begin{align*}
D_- &= \left( \begin{array}{c} \ell^2_- \\ 0 \end{array} \right), \\
D_+ &= \left( \begin{array}{c} 0 \\ \ell^2_+ \end{array} \right).
\end{align*}

Therefore $H_p$ is converted to the form

\begin{equation}
\tilde{H}_p = \left\{ f_n = \left( \begin{array}{c} f^1_n \\ f^2_n \end{array} \right) \in \ell^2_\mathbb{Z} \left( \begin{array}{cc} I \\ S_\Delta(n) \end{array} \right) ; f^1_n + S^*_\Delta(n)f^2_n \in \ell^2_+ , f^1_n + f^2_n \in \ell^2_- \right\}
\end{equation}

and a “semigroup” $\hat{T}_t$ will have the form

\begin{equation}
\tilde{T}_t(n)f_n = P_{\tilde{H}_p} e^{-itn}f_n.
\end{equation}

Thus the following theorem is proved.

Theorem 2.6. The minimal unitary dilatation $U_t$ (34) in $\mathcal{H}$ (32) of the family of operators $T_t$ (31) with a scattering operator $A$ of collegation $\Delta$ (1) is unitary equivalent to $\hat{U}_t(n)$ (52) in the space $\ell^2_\mathbb{Z} \left( \begin{array}{cc} I \\ S_\Delta(n) \end{array} \right)$, and the family $T_t$ (31) is unitary equivalent to $\tilde{T}_t(n)$ (54) in the space $\tilde{H}_p$.

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