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Archivum Mathematicum, Vol. 39 (2003), No. 3, 201--208

Persistent URL: http://dml.cz/dmlcz/107867

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AN EXTENSION OF THE METHOD OF QUASILINEARIZATION

TADEUSZ JANKOWSKI

Abstract. The method of quasilinearization is a well-known technique for obtaining approximate solutions of nonlinear differential equations. This method has recently been generalized and extended using less restrictive assumptions so as to apply to a larger class of differential equations. In this paper, we use this technique to nonlinear differential problems.

1. Introduction

Let $y_0, z_0 \in C^1(J, \mathbb{R})$ with $y_0(t) \leq z_0(t)$ on $J$ and define the following sets

$$\bar{\Omega} = \{(t, u) : y_0(t) \leq u \leq z_0(t), \ t \in J\},$$
$$\Omega = \{(t, u, v) : y_0(t) \leq u \leq z_0(t), \ y_0(t) \leq v \leq z_0(t), \ t \in J\}.$$ 

In this paper, we consider the following initial value problem

$$x'(t) = f(t, x(t)), \quad t \in J = [0, b], \ x(0) = k_0,$$

where $f \in C(\bar{\Omega}, \mathbb{R}), k_0 \in \mathbb{R}$ are given. If we replace $f$ by the sum $[f = g_1 + g_2]$ of convex and concave functions, then corresponding monotone sequences converge quadratically to the unique solution of problem (1) (see [6,8]). In this paper we will generalize this result. Assume that $f$ has the splitting $f(t, x) = F(t, x, x)$, where $F \in C(\Omega, \mathbb{R})$. Then problem (1) takes the form

$$x'(t) = F(t, x(t), x(t)), \quad t \in J, \ x(0) = k_0.$$

2000 Mathematics Subject Classification: 34A45.

Key words and phrases: quasilinearization, monotone iterations, quadratic convergence.

Received August 15, 2001.
2. Main results

A function $v \in C^1(J, \mathbb{R})$ is said to be a lower solution of problem (2) if

$$v'(t) \leq F(t, v(t), v(t)),$$ $t \in J$, $v(0) \leq k_0,$

and an upper solution of (2) if the inequalities are reversed.

**Theorem 1.** Assume that:

1° $y_0, z_0 \in C^1(J, \mathbb{R})$ are lower and upper solutions of problem (2), respectively, such that $y_0(t) \leq z_0(t)$ on $J$,

2° $F, F_x, F_y, F_{xx}, F_{xy}, F_{yy} \in C(\Omega, \mathbb{R})$ and

$$F_{xx}(t, x, y) \geq 0, \ F_{xy}(t, x, y) \leq 0, \ F_{yy}(t, x, y) \leq 0 \text{ for } (t, x, y) \in \Omega.$$

Then there exist monotone sequences $\{y_n\}, \{z_n\}$ which converge uniformly to the unique solution $x$ of (2) on $J$, and the convergence is quadratic.

**Proof.** The above assumptions guarantee that (2) has exactly one solution on $\Omega$.

Observe that $2°$ implies that $F_x$ is nondecreasing in the second variable, $F_x$ is nonincreasing in the third variable and $F_y$ is nonincreasing in the last two variables. Denote this property by (A).

Let us construct the elements of sequences $\{y_n\}, \{z_n\}$ by

$$y_{n+1}'(t) = F(t, y_n, y_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][y_{n+1}(t) - y_n(t)], \quad y_{n+1}(0) = k_0,$$

$$z_{n+1}'(t) = F(t, z_n, z_n) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][z_{n+1}(t) - z_n(t)], \quad z_{n+1}(0) = k_0$$

for $n = 0, 1, \cdots$. Note that the above sequences are well defined.

Indeed, $y_0(t) \leq z_0(t)$ on $J$, by $1°$. We shall show that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t) \text{ on } J.$$  

Put $p = y_0 - y_1$ on $J$. Then

$$p'(t) \leq F(t, y_0, y_0) - F(t, y_0, y_0) - [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][y_1(t) - y_0(t)]$$

$$= [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t).$$

Hence $p(t) \leq 0$ on $J$, since $p(0) \leq 0$, showing that $y_0(t) \leq y_1(t)$ on $J$. Note that if we put $p = z_1 - z_0$ on $J$, then

$$p'(t) \leq F(t, z_0, z_0) + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)][z_1(t) - z_0(t)] - F(t, z_0, z_0)$$

$$= [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)]p(t), \quad \text{and } p(0) \leq 0.$$
so \(z_1(t) \leq z_0(t)\) on \(J\). Next, we let \(p = y_1 - z_1\) on \(J\), so \(p(0) = 0\). By the mean value theorem and property (A), we have

\[
p'(t) = F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, y_0, y_0) - F(t, y_1, y_1) + F(t, y_1, y_1) + F(t, y_1, y_1)
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] [y_1(t) - y_0(t) - z_1(t) + z_0(t)]
\]

\[
= [F_x(t, \xi, y_0) + F_y(t, z_0, \sigma)] [y_0(t) - z_0(t)]
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, y_0, y_0)] [y_1(t) - y_0(t) - z_1(t) + z_0(t)]
\]

\[
\leq [F_x(t, y_0, z_0) - F_x(t, y_0, y_0)] [z_0(t) - y_0(t)]
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] p(t)
\]

\[
\leq [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] p(t)
\]

where \(y_0(t) < \xi(t), \sigma(t) < z_0(t)\) on \(J\). As the result we get \(p(t) \leq 0\) on \(J\), so \(y_1(y) \leq z_1(t)\) on \(J\). It proves that (3) holds.

Now we prove that \(y_1, z_1\) are lower and upper solutions of (2), respectively. The mean value theorem and property (A) yield

\[
y_1'(t) = F(t, y_0, y_0) - F(t, y_1, y_0) + F(t, y_1, y_0) - F(t, y_1, y_1) + F(t, y_1, y_1)
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] [y_1(t) - y_0(t)]
\]

\[
= [F_x(t, \xi, y_0) + F_y(t, y_1, \sigma)] [y_0(t) - y_1(t)] + F(t, y_1, y_1)
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] [y_1(t) - y_0(t)]
\]

\[
\leq [F_x(t, y_0, z_0) - F_x(t, y_0, y_0)] [y_1(t) - y_0(t)]
\]

\[
\quad + [F_y(t, y_1, y_0) + F_y(t, y_1, y_0)] [y_1(t) - y_0(t)]
\]

\[
\quad + F(t, y_1, y_1) \leq F(t, y_1, y_1),
\]

where \(y_0(t) < \xi(t), \sigma_1(t) < y_1(t)\) on \(J\). Similarly, we get

\[
z_1'(t) = F(t, z_1, z_1) + F(t, z_0, z_0) - F(t, z_1, z_0) + F(t, z_1, z_0) - F(t, z_1, z_1)
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] [z_1(t) - z_0(t)]
\]

\[
= F(t, z_1, z_1) + [F_x(t, \xi_2, z_0) + F_y(t, z_1, \sigma_2)] [z_0(t) - z_1(t)]
\]

\[
\quad + [F_x(t, y_0, z_0) + F_y(t, z_0, z_0)] [z_1(t) - z_0(t)]
\]

\[
\geq F(t, z_1, z_1) + [F_x(t, z_1, z_0) - F_x(t, y_0, z_0)] + F_y(t, z_1, z_0)
\]

\[
\quad - F_y(t, z_0, z_0)] [z_0(t) - z_1(t)] \geq F(t, z_1, z_1),
\]

where \(z_1(t) < \xi_2(t), \sigma_2(t) < z_0(t)\) on \(J\). The above proves that \(y_1, z_1\) are lower and upper solutions of (2).

Let us assume that

\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_{k-1}(t) \leq y_k(t) \leq z_k(t) \leq z_{k-1}(t) \leq \cdots \leq z_1(t) \leq z_0(t),
\]

\[
t \in J,
\]

and let \(y_k, z_k\) be lower and upper solutions of problem (2) for some \(k \geq 1\). We shall prove that:

\[
y_k(t) \leq y_{k+1}(t) \leq z_{k+1}(t) \leq z_k(t), \quad t \in J.
\]

(4)
Let \( p = y_k - y_{k+1} \) on \( J \), so \( p(0) = 0 \). Using the mean value theorem, property (A) and the fact that \( y_k \) is a lower solution of problem (2), we obtain

\[
p'(t) \leq F(t, y_k, y_k) - F(t, y_k, y_k) - [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t)]
\]

\[
= [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t).
\]

Hence \( p(t) \leq 0 \), so \( y_k(t) \leq y_{k+1}(t) \) on \( J \). Similarly, we can show that \( z_k(t) \leq z_{k+1}(t) \) on \( J \).

Now, if \( p = y_{k+1} - z_{k+1} \) on \( J \), then

\[
p'(t) = F(t, y_k, y_k) - F(t, z_k, y_k) + F(t, z_k, y_k) - F(t, z_k, z_k)
\]

\[
+ [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)]
\]

\[
= [F_x(t, \xi, y_k) + F_y(t, z_k, \sigma)][y_k(t) - z_k(t)]
\]

\[
+ [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)][y_{k+1}(t) - y_k(t) - z_{k+1}(t) + z_k(t)]
\]

\[
\leq [F_x(t, y_k, z_k) + F_y(t, y_k, z_k)][z_k(t) - y_k(t)]
\]

\[
+ [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t)
\]

\[
\leq [F_x(t, y_k, z_k) + F_y(t, z_k, z_k)]p(t)
\]

with \( y_k(t) < \xi(t), \sigma(t) < z_k(t) \). It proves that \( y_{k+1}(t) \leq z_{k+1}(t) \) on \( J \), so relation (4) holds.

Hence, by induction, we have

\[
y_0(0) \leq y_1(0) \leq \cdots \leq y_n(0) \leq z_n(t) \leq \cdots \leq z_1(t) \leq y_0(t), \quad t \in J,
\]

for all \( n \). Employing standard techniques [5], it can be shown that the sequences \( \{y_n\}, \{z_n\} \) converge uniformly and monotonically to the unique solution \( x \) of problem (2).

We shall next show the convergence of \( y_n, z_n \) to the unique solution \( x \) of problem (2) is quadratic. For this purpose, we consider

\[
p_{n+1} = x - y_{n+1} \geq 0, \quad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on} \quad J,
\]

and note that \( p_{n+1}(0) = q_{n+1}(0) = 0 \) for \( n \geq 0 \). Using the mean value theorem and property (A), we get

\[
p'_{n+1}(t) = F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n)
\]

\[
- [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][y_{n+1}(t) - x(t) + x(t) - y_n(t)]
\]

\[
= [F_x(t, \xi_1, x) + F_y(t, y_n, \sigma_1)]p_n(t)
\]

\[
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][p_{n+1}(t) - p_n(t)]
\]

\[
\leq [F_x(t, x, x) - F_x(t, y_n, x) + F_x(t, y_n, x) - F_x(t, y_n, z_n)
\]

\[
+ F_y(t, y_n, y_n) - F_y(t, z_n, y_n) + F_y(t, z_n, y_n) - F_y(t, z_n, z_n)]p_n(t)
\]

\[
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t)
\]

\[
= \{F_{xx}(t, \xi_2, x)p_n(t) - F_{xy}(t, y_n, \sigma_2)q_n(t) - F_{yx}(t, \xi_3, y_n)[z_n(t) - y_n(t)]
\]

\[
- F_{yy}(t, z_n, \sigma_3)[z_n(t) - y_n(t)]\}p_n(t)
\]

\[
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)]p_{n+1}(t),
\]
where \( y_n(t) < \bar{\xi}_1(t), \bar{\xi}_2(t), \bar{\sigma}_1(t) < x(t), x(t) < \bar{\sigma}_2(t) < z_n(t), y_n(t) < \bar{\xi}_3(t), \bar{\sigma}_3(t) < z_n(t) \) on \( J \). Thus we obtain

\[
p_n'(t) \leq \{ A_1 p_n(t) + A_2 q_n(t) + [A_2 + A_3][q_n(t) + p_n(t)]\} p_n(t) + M p_{n+1}(t) \\
\leq M p_{n+1}(t) + B_1 p_n^2(t) + B_2 q_n^2(t),
\]

where

\[
|F_{xx}(t, u, v)| \leq A_1, \quad |F_{xy}(t, u, v)| \leq A_2, \quad |F_{yy}(t, u, v)| \leq A_3, \quad |F_x(t, u, v)| \leq M_1, \\
|F_y(t, u, v)| \leq M_2 \quad \text{on} \quad \Omega \quad \text{with} \quad M = M_1 + M_2, \quad B_1 = A_1 + 2A_2 + \frac{3}{2}A_3, \\
B_2 = A_2 + \frac{1}{2}A_3.
\]

Now, the differential inequality implies

\[
0 \leq p_{n+1}(t) \leq \int_0^t [B_1 p_n^2(s) + B_2 q_n^2(s)] \exp[M(t-s)] \, ds.
\]

This yields the following relation

\[
\max_{t \in J} |x(t) - y_{n+1}(t)| \leq a_1 \max_{t \in J} |x(t) - y_n(t)|^2 + a_2 \max_{t \in J} |x(t) - z_n(t)|^2,
\]

where \( a_i = B_i S, i = 1, 2 \) with

\[
S = \begin{cases} 
  b & \text{if} \quad M = 0, \\
  \frac{1}{M}[\exp(Mb) - 1] & \text{if} \quad M > 0.
\end{cases}
\]

Similarly, we find that

\[
q_n'(t) = F(t, z_n, z_n) - F(t, x, z_n) + F(t, x, z_n) - F(t, x, x) \\
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][q_n(t) - x(t) + x(t) - z_n(t)] \\
= [F_x(t, \bar{\xi}_4, z_n) + F_y(t, x, \bar{\sigma}_4)] q_n(t) \\
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)][q_n(t) - q_n(t)] \\
\leq [F_x(t, z_n, z_n) - F_x(t, y_n, z_n) + F_y(t, x, x) - F_y(t, z_n, x) \\
+ F_y(t, z_n, x) - F_y(t, z_n, z_n)] q_n(t) + [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)] q_n(t) \\
= \{ F_{xx}(t, \bar{\xi}_5, z_n)[z_n(t) - y_n(t)] \\
- F_{xy}(t, \bar{\xi}_6, x) q_n(t) - F_{yy}(t, z_n, \bar{\sigma}_5) q_n(t)\} q_n(t) \\
+ [F_x(t, y_n, z_n) + F_y(t, z_n, z_n)] q_n(t),
\]

where \( x(t) < \bar{\xi}_4(t), \bar{\xi}_6(t), \bar{\sigma}_4(t), \bar{\sigma}_5(t) < z_n(t), y_n(t) < \bar{\xi}_5(t) < z_n(t) \) on \( J \). Hence, we get

\[
q_n'(t) \leq \{ A_1 q_n(t) + p_n(t) + A_2 q_n(t) + A_3 q_n(t)\} q_n(t) + M q_{n+1}(t), \\
\leq M q_{n+1}(t) + B_1 p_n^2(t) + B_2 q_n^2(t),
\]
where
\[ \tilde{B}_1 = \frac{1}{2} A_1, \quad \tilde{B}_2 = \frac{3}{2} A_1 + A_2 + A_3. \]

Now, the last differential inequality implies
\[ q_{n+1}(t) \leq [\tilde{B}_1 \max_{s \in J} q_n^2(s) + \tilde{B}_2 \max_{s \in J} q_n^2(s)]S, \quad t \in J \]
or
\[ \max_{t \in J} |x(t) - z_{n+1}(t)| \leq \bar{a}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \bar{a}_2 \max_{t \in J} |x(t) - z_n(t)|^2 \]
with \( \bar{a}_i = \tilde{B}_i S, \ i = 1, 2. \)

The proof is complete. \( \square \)

**Remark 1.** Let \( f = h + g, \) and \( h, h_x, h_{xx}, g, g_x, g_{xx} \in C(\Omega_1, \mathbb{R}) \) for \( \Omega_1 = \{(t, u) : t \in J, y_0(t) \leq u \leq z_0(t)\}. \) Put \( F(t, x, y) = h(t, x) + g(t, y). \) Indeed, \( F(t, x, x) = f(t, x) \) and \( F_{xx}(t, x, y) = h_{xx}(t, x), \ F_{xy}(t, x, y) = F_{yx}(t, x, y) = 0, \ F_{yy}(t, x, y) = g_{yy}(t, y) \) for \( G = g + \Psi. \) Indeed, \( F(t, x, x) = f(t, x) \) and \( F_{xx}(t, x, y) = h_{xx}(t, x) - \Psi_{xx}(t, x), \ F_{xy}(t, x, y) = F_{yx}(t, x, y) = 0, \ F_{yy}(t, x, y) = G_{yy}(t, y) - \Psi_{yy}(t, y). \) If assumptions of Theorem 1.4.3[8] hold \( (H_{xx} \geq 0, \ \Psi_{xx} \leq 0, \ G_{yy} \leq 0, \ \Phi_{yy} \geq 0) \) then Theorem 1 is satisfied (see also a result of [6] for \( g = \Psi = 0, \ \Phi(t, x) = Mx^2, \ M > 0). \)

**Theorem 2.** Assume that
(i) condition 1° of Theorem 1 holds,
(ii) \( F, F_x, F_y, F_{xx}, F_{xy}, F_{yx}, F_{yy} \in C(\Omega, \mathbb{R}) \) and
\[ F_{xx}(t, x, y) \geq 0, \quad F_{xy}(t, x, y) \geq 0, \quad F_{yy}(t, x, y) \leq 0 \quad \text{for} \quad (t, x, y) \in \Omega. \]

Then the conclusion of Theorem 1 remains valid.

**Proof.** Note that, in view of (ii), \( F_x \) is nondecreasing in the last two variables, \( F_y \) is nondecreasing in the second variable, and \( F_{xy} \) is nonincreasing in the third one. Denote this property by (B).

We construct the monotone sequences \( \{y_n\}, \{z_n\} \) by formulas:
\[
y_{n+1}'(t) = F(t, y_n, y_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][y_{n+1}(t) - y_n(t)], \quad y_{n+1}(0) = k_0,
\]
\[
z_{n+1}'(t) = F(t, z_n, z_n) + [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][z_{n+1}(t) - z_n(t)], \quad z_{n+1}(0) = k_0
\]
for \( n = 0, 1, \ldots \).
Let $p = y_0 - y_1$ on $J$. Then

$$p'(t) \leq F(t, y_0, y_0) - F(t, y_0, y_0) - [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)][y_1(t) - y_0(t)]$$

$$= [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t), \quad \text{and } p(0) \leq 0.$$ 

Hence $p(t) \leq 0$ on $J$, showing that $y_0(t) \leq y_1(t)$ on $J$. Similarly, we can show that $z_1(t) \leq z_0(t)$ on $J$. If we now put $p = y_1 - z_1$ on $J$, then the mean value theorem and property (B), we have

$$p'(t) = F(t, y_0, y_0) - F(t, z_0, y_0) + F(t, z_0, y_0) - F(t, z_0, z_0)$$

$$= [F_x(t, \xi, y_0) + F_y(t, z_0, \sigma)][y_0(t) - z_0(t)]$$

$$= [F_y(t, y_0, z_0) - F_y(t, z_0, z_0)][z_0(t) - y_0(t)]$$

$$\leq [F_y(t, y_0, z_0) - F_y(t, z_0, z_0)][z_0(t) - y_0(t)]$$

$$= [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t)$$

$$\leq [F_x(t, y_0, y_0) + F_y(t, y_0, z_0)]p(t), \quad p(0) = 0$$

with $y_0(t) < \xi(t)$, $\sigma(t) < z_0(t)$ on $J$. Hence $y_1(t) \leq z_1(t)$ on $J$, and as a result, we obtain

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_1(t) \leq z_0(t) \quad \text{on } J.$$ 

Continuing this process successively, by induction, we get

$$y_0(t) \leq y_1(t) \leq \cdots \leq y_n(t) \leq z_n(t) \leq \cdots \leq z_1(t) \leq z_0(t), \quad t \in J,$$

for all $n$. Indeed, the sequences $\{y_n\}$, $\{z_n\}$ converge uniformly and monotonically to the unique solution $x$ of problem (2). Now, we are in a position to show that this convergence is quadratic.

Let

$$p_{n+1} = x - y_{n+1} \geq 0, \quad q_{n+1} = z_{n+1} - x \geq 0 \quad \text{on } J.$$ 

Hence $p_{n+1}(0) = q_{n+1}(0) = 0$. The mean value theorem and property (B) yield

$$p'_{n+1}(t) = F(t, x, x) - F(t, y_n, x) + F(t, y_n, x) - F(t, y_n, y_n)$$

$$= [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)][y_{n+1}(t) - x(t) + x(t) - y_n(t)]$$

$$= [F_x(t, \xi_1, x) + F_y(t, y_n, \sigma_1)]p_n(t)$$

$$+ [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t)$$

$$\leq [F_x(t, x, x) - F_x(t, y_n, x) + F_x(t, y_n, x) - F_x(t, y_n, y_n)$$

$$+ F_y(t, y_n, y_n) - F_y(t, y_n, z_n)]p_n(t)$$

$$+ [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t)$$

$$= [F_x(t, \xi_2, x)]p_n(t) + F_{xy}(t, y_n, \sigma_2)p_n(t)$$

$$- F_{yy}(t, y_n, \sigma_3)[z_n(t) - y_n(t)]p_n(t)$$

$$+ [F_x(t, y_n, y_n) + F_y(t, y_n, z_n)]p_{n+1}(t),$$
where $y_n(t) < \xi_1(t)$, $\xi_2(t), \sigma_1(t), \sigma_2(t) < x(t)$, $y_n(t) < \sigma_3(t) < z_n(t)$ on $J$. Thus we obtain
\[
p_{n+1}(t) \leq \{ (A_1 + A_2)p_n(t) + A_3[q_n(t) + p_n(t)] \} p_n(t) + M p_{n+1}(t)
\leq M p_{n+1}(t) + D_1 p_n^2(t) + D_2 q_n^2(t),
\]
where $D_1 = A_1 + A_2 + \frac{3}{2} A_3$, $D_2 = \frac{1}{2} A_3$. Hence, we get
\[0 \leq p_{n+1}(t) \leq \int_0^t \left[ D_1 p_n^2(s) + D_2 q_n^2(s) \right] \exp[M(t-s)] ds,
\]
and it yields the relation
\[
\max_{t \in J} |x(t) - y_{n+1}(t)| \leq d_1 \max_{t \in J} |x(t) - y_n(t)|^2 + d_2 \max_{t \in J} |x(t) - z_n(t)|^2,
\]
where $d_i = D_i S$, $i = 1, 2$.

By the similar argument, we can show that
\[
\max_{t \in J} |x(t) - z_{n+1}(t)| \leq \tilde{d}_1 \max_{t \in J} |x(t) - y_n(t)|^2 + \tilde{d}_2 \max_{t \in J} |x(t) - z_n(t)|^2,
\]
with $\tilde{d}_i = \tilde{D}_i S$, $i = 1, 2$, for $\tilde{D}_1 = \frac{1}{2} A_1 + A_2$, $\tilde{D}_2 = \frac{3}{2} A_1 + 2 A_2 + A_3$.

This ends the proof. \hfill \Box

References