Suthep Suantai
On the $H$-property of some Banach sequence spaces

Archivum Mathematicum, Vol. 39 (2003), No. 4, 309--316

Persistent URL: http://dml.cz/dmlcz/107879

Terms of use:

© Masaryk University, 2003

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ON THE H–PROPERTY OF SOME
BANACH SEQUENCE SPACES

SUTHEP SUANTAI

ABSTRACT. In this paper we define a generalized Cesàro sequence space \( \text{ces}(p) \) and consider it equipped with the Luxemburg norm under which it is a Banach space, and we show that the space \( \text{ces}(p) \) possesses property (H) and property (G), and it is rotund, where \( p = (p_k) \) is a bounded sequence of positive real numbers with \( p_k > 1 \) for all \( k \in \mathbb{N} \).

1. Preliminaries

For a Banach space \( X \), we denote by \( S(X) \) and \( B(X) \) the unit sphere and unit ball of \( X \), respectively. A point \( x_0 \in S(X) \) is called

a) an extreme point if for every \( x, y \in S(X) \) the equality \( 2x_0 = x + y \) implies \( x = y \);

b) an H-point if for any sequence \( (x_n) \) in \( X \) such that \( \|x_n\| \to 1 \) as \( n \to \infty \), the weak convergence of \( (x_n) \) to \( x_0 \) (write \( x_n \overset{w}{\to} x_0 \)) implies that \( \|x_n - x\| \to 0 \) as \( n \to \infty \);

c) a denting point if for every \( \epsilon > 0 \), \( x_0 \notin \overline{\text{conv}}\{B(X) \setminus (x_0 + \epsilon B(X))\} \).

A Banach space \( X \) is said to be rotund (R), if every point of \( S(X) \) is an extreme point.

A Banach space \( X \) is said to possess property (H) (property (G)) provided every point of \( S(X) \) is H-point (denting point).

For these geometric notions and their role in mathematics we refer to the monographs [1], [2], [6] and [13]. Some of them were studied for Orlicz spaces in [3], [7], [8], [9] and [114].

Let us denote by \( l^0 \) the space of all real sequences. For \( 1 \leq p < \infty \), the Cesàro sequence space \( \text{ces}_p \) (for short) is defined by

\[
\text{ces}_p = \left\{ x \in l^0 : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p < \infty \right\}
\]

2000 Mathematics Subject Classification: 46E30, 46E40, 46B20.

Key words and phrases: H-property, property (G), Cesàro sequence spaces, Luxemburg norm.

Received November 13, 2001.
equipped with the norm
\[
\|x\| = \left( \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^\frac{1}{p}
\]

This space was introduced by J.S. Shue [16]. It is useful in the theory of matrix operator and others (see [10] and [12]). Some geometric properties of the Cesàro sequence space \( \text{ces}_p \) were studied by many mathematicians. It is known that \( \text{ces}_p \) is LUR and posses property (H) (see [12]). Y.A. Cui and H. Hudzik [14] proved that \( \text{ces}_p \) has the Banach-Saks of type \( p \) if \( p > 1 \), and it was shown in [5] that \( \text{ces}_p \) has property \( (\beta) \).

Now, let \( p = (p_k) \) be a sequence of positive real numbers with \( p_k \geq 1 \) for all \( k \in \mathbb{N} \). The Nakano sequence space \( l(p) \) is defined by
\[
l(p) = \{ x \in l^0 : \sigma(\lambda x) < \infty \text{ for some } \lambda > 0 \},
\]
where \( \sigma(x) = \sum_{i=1}^{\infty} |x(i)|^{p_i} \). We consider the space \( l(p) \) equipped with the norm
\[
\|x\| = \inf \left\{ \lambda > 0 : \sigma \left( \frac{x}{\lambda} \right) \leq 1 \right\},
\]
under which it is a Banach space. If \( p = (p_k) \) is bounded, we have
\[
l(p) = \left\{ x \in l^0 : \sum_{i=1}^{\infty} |x(i)|^{p_i} < \infty \right\}.
\]

Several geometric properties of \( l(p) \) were studied in [1] and [4].

The Cesàro sequence space \( \text{ces}(p) \) is defined by
\[
\text{ces}(p) = \{ x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0 \},
\]
where \( \varrho(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^{p_n} \). We consider the space \( \text{ces}(p) \) equipped with the so-called Luxemburg norm
\[
\|x\| = \inf \left\{ \lambda > 0 : \varrho \left( \frac{x}{\lambda} \right) \leq 1 \right\}
\]
under which it is a Banach space. If \( p = (p_k) \) is bounded, then we have
\[
\text{ces}(p) = \left\{ x = x(i) : \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^{p_n} < \infty \right\}.
\]

W. Sanhan [15] proved that \( \text{ces}(p) \) is nonsquare when \( p_k > 1 \) for all \( k \in \mathbb{N} \). In this paper, we show that the Cesàro sequence space \( \text{ces}(p) \) equipped with the Luxemburg norm is rotund (R) and posses property (H) and property (G) when \( p = (p_k) \) is bounded with \( p_k > 1 \) for all \( k \in \mathbb{N} \).

Throughout this paper we assume that \( p = (p_k) \) is bounded with \( p_k > 1 \) for all \( k \in \mathbb{N} \), and \( M = \sup_k p_k \).
2. Main Results

We begin with giving some basic properties of modular on the space ces\((p)\).

**Proposition 2.1.** The functional \(q\) on the Cesàro sequence space ces\((p)\) is a convex modular.

**Proof.** It is obvious that \(q(x) = 0 \iff x = 0\) and \(q(\alpha x) = q(x)\) for all scalar \(\alpha\) with \(|\alpha| = 1\). If \(x, y \in \text{ces}(p)\) and \(\alpha \geq 0, \beta \geq 0\) with \(\alpha + \beta = 1\), by the convexity of the function \(t \to |t|^p_k\) for every \(k \in \mathbb{N}\), we have

\[
q(\alpha x + \beta y) = \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |\alpha x(i) + \beta y(i)| \right)^{p_k} \\
\leq \sum_{k=1}^{\infty} \left( \alpha \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right) + \beta \left( \frac{1}{k} \sum_{i=1}^{k} |y(i)| \right) \right)^{p_k} \\
\leq \alpha \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \beta \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |y(i)| \right)^{p_k} \\
= \alpha q(x) + \beta q(y).
\]

**Proposition 2.2.** For \(x \in \text{ces}(p)\), the modular \(q\) on ces\((p)\) satisfies the following properties:

(i) if \(0 < a < 1\), then \(a^M q\left(\frac{x}{a}\right) \leq q(x)\) and \(q(ax) \leq a q(x)\),

(ii) if \(a \geq 1\), then \(q(x) \leq a^M q\left(\frac{x}{a}\right)\),

(iii) if \(a \geq 1\), then \(q(x) \leq a q(x) \leq q(ax)\).

**Proof.** It is obvious that (iii) is satisfied by the convexity of \(q\). It remains to prove (i) and (ii).

For \(0 < a < 1\), we have

\[
q(x) = \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} = \sum_{k=1}^{\infty} \left( \frac{a}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \\
= \sum_{k=1}^{\infty} a^{p_k} \left( \frac{1}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \geq \sum_{k=1}^{\infty} a^M \left( \frac{1}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \\
= a^M \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} \left| \frac{x(i)}{a} \right| \right)^{p_k} \leq a^M q\left(\frac{x}{a}\right),
\]

and it implies by the convexity of \(q\) that \(q(ax) \leq a q(x)\), hence (i) is satisfied.
Now, suppose that $a \geq 1$. Then we have

$$\varrho(x) = \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} = \sum_{k=1}^{\infty} a^{p_k} \left( \frac{1}{k} \sum_{i=1}^{k} \frac{|x(i)|}{a} \right)^{p_k} \leq a^M \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} \frac{|x(i)|}{a} \right)^{p_k} = a^M \varrho \left( \frac{x}{a} \right).$$

So (ii) is obtained. \(\square\)

Next, we give some relationships between the modular $\varrho$ and the Luxemburg norm on $\text{ces} (p)$.

**Proposition 2.3.** For any $x \in \text{ces} (p)$, we have

(i) if $\|x\| < 1$, then $\varrho(x) \leq \|x\|$, 
(ii) if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$, 
(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$, 
(iv) $\|x\| < 1$ if and only if $\varrho(x) < 1$, 
(v) $\|x\| > 1$ if and only if $\varrho(x) > 1$, 
(vi) if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$, and 
(vii) if $a \geq 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.

**Proof.** (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \varepsilon < 1$. By definition of $\|\cdot\|$, there exists $\lambda > 0$ such that $\|x\| + \varepsilon > \lambda$ and $\varrho (\frac{x}{\lambda}) \leq 1$. From Proposition 2.2 (i) and (iii), we have

$$\varrho(x) \leq \varrho \left( (\|x\| + \varepsilon) \frac{x}{\lambda} \right) = \varrho \left( \left( \|x\| + \varepsilon \right) \frac{x}{\lambda} \right) \leq \left( \|x\| + \varepsilon \right) \varrho \left( \frac{x}{\lambda} \right) \leq \|x\| + \varepsilon,$$

which implies that $\varrho(x) \leq \|x\|$, so (i) is satisfied.

(ii) Let $\varepsilon > 0$ be such that $0 < \varepsilon < \frac{\|x\|-1}{\|x\|}$, then $1 < (1 - \varepsilon) \|x\| < \|x\|$. By definition of $\|\cdot\|$ and by Proposition 2.2 (i), we have

$$1 < \varrho \left( \frac{x}{(1 - \varepsilon) \|x\|} \right) \leq \frac{1}{(1 - \varepsilon) \|x\|} \varrho(x),$$

so $(1 - \varepsilon) \|x\| < \varrho(x)$ for all $\varepsilon \in (0, \frac{\|x\| - 1}{\|x\|})$. This implies that $\|x\| \leq \varrho(x)$, hence (ii) is obtained.

(iii) Assume that $\|x\| = 1$. By definition of $\|x\|$, we have that for $\varepsilon > 0$, there exists $\lambda > 0$ such that $1 + \varepsilon > \lambda > \|x\|$ and $\varrho (\frac{x}{\lambda}) \leq 1$. From Proposition 2.2 (ii), we have $\varrho(x) \leq \lambda^M \varrho \left( \frac{x}{\lambda} \right) \leq \lambda^M < (1 + \varepsilon)^M$, so $(\varrho(x))^\frac{1}{M} < 1 + \varepsilon$ for all $\varepsilon > 0$, which implies $\varrho(x) \leq 1$. If $\varrho(x) < 1$, then we can choose $a \in (0,1)$ such that
Lemma 2.5. Suppose we have (1) as \( n \to \infty \), we obtain that \( \|x\| < a < 1 \), which is a contradiction. Therefore \( \varrho(x) = 1 \).

On the other hand, assume that \( \varrho(x) = 1 \). Then \( \|x\| < 1 \), we have by (i) that \( \varrho(x) \leq \|x\| < 1 \), which contradicts our assumption. Therefore \( \|x\| = 1 \).

(iv) follows directly from (i) and (iii).

(v) follows from (iii) and (iv).

(vi) Suppose \( 0 < a < 1 \) and \( \|x\| > a \). Then \( \|\frac{x}{a}\| > 1 \). By (v), we have \( \varrho\left(\frac{x}{a}\right) > 1 \). Hence, by Proposition 2.2 (i), we obtain that \( \varrho(x) \geq a^M \varrho\left(\frac{x}{a}\right) > a^M \).

(vii) Suppose \( a \geq 1 \) and \( \|x\| < a \). Then \( \|\frac{x}{a}\| < 1 \). By (iv), we have \( \varrho\left(\frac{x}{a}\right) < 1 \). If \( a = 1 \), it is obvious that \( \varrho(x) < 1 = a^M \). If \( a > 1 \), then, by Proposition 2.2 (ii), we obtain that \( \varrho(x) \leq a^M \varrho\left(\frac{x}{a}\right) < a^M \).

Proposition 2.4. Let \( (x_n) \) be a sequence in \( \text{ces}(p) \).

(i) If \( \|x_n\| \to 1 \) as \( n \to \infty \), then \( \varrho(x_n) \to 1 \) as \( n \to \infty \).

(ii) If \( \varrho(x_n) \to 0 \) as \( n \to \infty \), then \( \|x_n\| \to 0 \) as \( n \to \infty \).

Proof. (i) Suppose \( \|x_n\| \to 1 \) as \( n \to \infty \). Let \( \epsilon \in (0,1) \). Then there exists \( N \in \mathbb{N} \) such that \( 1 - \epsilon < \|x_n\| < 1 + \epsilon \) for all \( n \geq N \). By Proposition 2.3 (vi) and (vii), we have \( (1 - \epsilon)^M < \varrho(x_n) < (1 + \epsilon)^M \) for all \( n \geq N \), which implies that \( \varrho(x_n) \to 1 \) as \( n \to \infty \).

(ii) Suppose \( \|x_n\| \not\to 0 \) as \( n \to \infty \). Then there is an \( \epsilon \in (0,1) \) and a subsequence \( (x_{n_k}) \) of \( (x_n) \) such that \( \|x_{n_k}\| > \epsilon \) for all \( k \in \mathbb{N} \). By Proposition 2.3 (vi), we have \( \varrho(x_{n_k}) > \epsilon^M \) for all \( k \in \mathbb{N} \). This implies \( \varrho(x_n) \not\to 0 \) as \( n \to \infty \).

Next, we shall show that \( \text{ces}(p) \) has the property (H). To do this, we need a lemma.

Lemma 2.5. Let \( x \in \text{ces}(p) \) and \( (x_n) \subseteq \text{ces}(p) \). If \( \varrho(x_n) \to \rho(x) \) as \( n \to \infty \) and \( x_n(i) \to x(i) \) as \( n \to \infty \) for all \( i \in \mathbb{N} \), then \( x_n \to x \) as \( n \to \infty \).

Proof. Let \( \epsilon > 0 \) be given. Since \( \rho(x) = \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} < \infty \), there is \( k_0 \in \mathbb{N} \) such that

\[
\sum_{k=k_0+1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} < \frac{\epsilon}{3} \frac{1}{2^{M+1}}.
\]

Since \( \rho(x_n) - \sum_{k=1}^{k_0} \left( \frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} \to \rho(x) - \sum_{k=1}^{k_0} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} \) as \( n \to \infty \) and \( x_n(i) \to x(i) \) as \( n \to \infty \) for all \( i \in \mathbb{N} \), there is \( n_0 \in \mathbb{N} \) such that

\[
\rho(x_n) - \sum_{k=1}^{k_0} \left( \frac{1}{k} \sum_{i=1}^{k} |x_n(i)| \right)^{p_k} < \rho(x) - \sum_{k=1}^{k_0} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^{p_k} + \frac{\epsilon}{3} \frac{1}{2^{M+1}}
\]

for all \( n \geq n_0 \), and

\[
\sum_{k=1}^{k_0} \left( \frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^{p_k} < \frac{\epsilon}{3}.
\]
for all $n \geq n_0$.

It follows from (2.1), (2.2) and (2.3) that for $n \geq n_0$,

$$\varrho(x_n - x) = \sum_{k=1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^p \geq \sum_{k=1}^{k_0} \left( \frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^p + \sum_{k=k_0+1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x_n(i) - x(i)| \right)^p$$

$$\leq \frac{\varepsilon}{3} + 2^M \left( \sum_{k=k_0+1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^p \right) + \sum_{k=k_0+1}^{\infty} \left( \frac{1}{k} \sum_{i=1}^{k} |x(i)| \right)^p$$

This shows that $\varrho(x_n - x) \to 0$ as $n \to \infty$. Hence, by Proposition 2.4 (ii), we have $\|x_n - x\| \to 0$ as $n \to \infty$.

**Theorem 2.6.** The space $\text{ces}(p)$ has the property $(H)$.

**Proof.** Let $x \in S(\text{ces}(p))$ and $(x_n) \subseteq \text{ces}(p)$ such that $\|x_n\| \to 1$ and $x_n \overset{w}{\rightharpoonup} x$ as $n \to \infty$. From Proposition 2.3 (iii), we have $\varrho(x) = 1$, so it follows from Proposition 2.4 (i) that $\varrho(x_n) \to \varrho(x)$ as $n \to \infty$. Since the mapping $p_i : \text{ces}(p) \to \mathbb{R}$, defined by $p_i(y) = y(i)$, is a continuous linear functional on $\text{ces}(p)$, it follows that $x_n(i) \to x(i)$ as $n \to \infty$ for all $i \in \mathbb{N}$. Thus, we obtain by Lemma 2.5 that $x_n \to x$ as $n \to \infty$. \hfill $\Box$

**Theorem 2.7.** The space $\text{ces}(p)$ is rotund.

**Proof.** Let $x \in S(\text{ces}(p))$ and $y, z \in B(\text{ces}(p))$ with $x = \frac{y + z}{2}$. By Proposition 2.3 and the convexity of $\varrho$ we have

$$1 = \varrho(x) \leq \frac{1}{2} \left( \varrho(y) + \varrho(z) \right) \leq \frac{1}{2} (1 + 1) = 1,$$
so that \( g(x) = \frac{1}{2}(g(y) + g(z)) = 1 \). This implies that

\[
(2.4) \quad \left( \frac{1}{k} \sum_{i=1}^{k} \left| \frac{y(i) + z(i)}{2} \right| \right)^{p_k} = \frac{1}{2} \left( \frac{1}{k} \sum_{i=1}^{k} |y(i)| \right)^{p_k} + \frac{1}{2} \left( \frac{1}{k} \sum_{i=1}^{k} |z(i)| \right)^{p_k}
\]

for all \( k \in \mathbb{N} \).

We shall show that \( y(i) = z(i) \) for all \( i \in \mathbb{N} \).

From (2.4), we have

\[
(2.5) \quad |x(1)|^{p_1} = \left| \frac{y(1) + z(1)}{2} \right|^{p_1} = \frac{1}{2} (|y(1)|^{p_1} + |z(1)|^{p_1}).
\]

Since the mapping \( t \to |t|^{p_1} \) is strictly convex, it implies by (2.5) that \( y(1) = z(1) \).

Now assume that \( y(i) = z(i) \) for all \( i = 1, 2, 3, \ldots, k-1 \). Then \( y(i) = z(i) = x(i) \) for all \( i = 1, 2, 3, \ldots, k-1 \). From (2.4), we have

\[
(2.6) \quad \left( \frac{1}{k} \sum_{i=1}^{k} \left| \frac{y(i) + z(i)}{2} \right| \right)^{p_k} = \frac{1}{k} \sum_{i=1}^{k} |y(i)|^{p_k} + \frac{1}{k} \sum_{i=1}^{k} |z(i)|^{p_k}
\]

By convexity of the mapping \( t \to |t|^{p_k} \), it implies that \( \frac{1}{k} \sum_{i=1}^{k} |y(i)| = \frac{1}{k} \sum_{i=1}^{k} |z(i)| \). Since \( y(i) = z(i) \) for all \( i = 1, 2, 3, \ldots, k-1 \), we get that

\[
(2.7) \quad |y(k)| = |z(k)|.
\]

If \( y(k) = 0 \), then we have \( z(k) = y(k) = 0 \). Suppose that \( y(k) \neq 0 \). Then \( z(k) \neq 0 \). If \( y(k)z(k) < 0 \), it follows from (2.7) that \( y(k) + z(k) = 0 \). This implies by (2.6) and (2.7) that

\[
\left( \frac{1}{k} \sum_{i=1}^{k-1} |x(i)| \right)^{p_k} = \left( \frac{1}{k} \left( \sum_{i=1}^{k-1} |x(i)| + |y(k)| \right) \right)^{p_k},
\]

which is a contradiction. Thus, we have \( y(k)z(k) > 0 \). This implies by (2.5) that \( y(k) = z(k) \). Thus, we have by induction that \( y(i) = z(i) \) for all \( i \in \mathbb{N} \), so \( y = z \).

Bor-Luh Lin, Pei-Kee Lin and S.L. Troyanski proved (cf. Theorem iii [11]) that element \( x \) in a bounded closed convex set \( K \) of a Banach space is a denting point of \( K \) iff \( x \) is an \( H \)-point of \( K \) and \( x \) is an extreme point of \( K \). Combining this result with our results (Theorem 2.6 and Theorem 2.7), we obtain the following result.
Corollary 2.8. The space $\text{ces}(p)$ has the property $(G)$.

For $1 < r < \infty$, let $p = (p_k)$ with $p_k = r$ for all $k \in \mathbb{N}$. We have that $\text{ces}_r = \text{ces}(p)$, so the following results are obtained directly from Theorem 2.6, Theorem 2.7 and Corollary 2.8, respectively.

Corollary 2.9. For $1 < r < \infty$, the Cesàro sequence space $\text{ces}_r$ has the property $(H)$.

Corollary 2.10. For $1 < r < \infty$, the Cesàro sequence space $\text{ces}_r$ is rotund.

Corollary 2.11. For $1 < r < \infty$, the Cesàro sequence space $\text{ces}_r$ has the property $(G)$.

Acknowledgements. The author would like to thank the Thailand Research Fund for the financial support.

References