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FINITENESS OF A CLASS OF RABINOWITSCH POLYNOMIALS

JAN-CHRISTOPH SCHLAGE-PUCHTA

Abstract. We prove that there are only finitely many positive integers \( m \) such that there is some integer \( t \) such that \(|n^2 + n - m|\) is 1 or a prime for all \( n \in [t + 1, t + \sqrt{m}] \), thus solving a problem of Byeon and Stark.

In 1913, G. Rabinowitsch [4] proved that for any positive integer \( m \) with square-free \( 4m - 1 \), the class number of \( \mathbb{Q}(\sqrt{1-4m}) \) is 1 if and only if \( n^2 + n + m \) is prime for all integers \( 0 \leq n \leq m - 3 \). Recently, D. Byeon and H. M. Stark [1] proved an analogue statement for real quadratic fields. The polynomial \( f_m(x) = x^2 + x - m \) is called a Rabinowitsch polynomial, if there is some integer \( t \) such that \(|f_m(n)|\) is 1 or a prime for all integral \( n \in [t + 1, t + \sqrt{m}] \). They proved the following theorem:

**Theorem 1.**
1. If \( f_m \) is Rabinowitsch, then one of the following equations hold:
   \[ m = 1, \quad m = 2, \quad m = p^2 \text{ for some odd prime } p, \quad m = t^2 + t \pm 1, \quad \text{or } m = t^2 + t \pm \frac{2t+1}{3}, \] where \( \frac{2t+1}{3} \) is an odd prime.
2. If \( f_m \) is Rabinowitsch, then \( \mathbb{Q}(\sqrt{4m+1}) \) has class number 1.
3. There are only finitely many \( m \) such that \( 4m + 1 \) is squarefree and that \( f_m \) is Rabinowitsch.

They asked whether the finiteness of \( m \) holds without the assumption on \( 4m+1 \). It is the aim of this note to show that this is indeed the case.

**Theorem 2.** There are only finitely many \( m \geq 0 \) such that \( f_m \) is Rabinowitsch.

For the proof write \( 4m + 1 = u^2D \) with \( D \) squarefree and \( u \) a positive integer. We distinguish three cases, namely \( D = 1 \), \( 1 < D < m^{1/12} \) and \( D \geq m^{1/12} \), and formulate each as a separate lemma. The first two cases are solved elementary, while the last one requires a slight extension of the argument in the case \( 4m + 1 \) squarefree given by Byeon and Stark.

**Lemma 1.** If \( f_m \) is Rabinowitsch and \( D = 1 \), then \( m = 2 \).

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Proof. We only deal with the case \( m = t^2 + t + \frac{2t+1}{3} \), the other cases are similar. Assume that \( D = 1 \), that is \( 4t^2 + \frac{20t}{3} + \frac{7}{3} = u^2 \). We have
\[
4t^2 + 4t + 1 < 4t^2 + \frac{20t}{3} + \frac{7}{3} < 4t^2 + 8t + 4
\]
that is, \( 2t + 1 < u < 2t + 2 \), which is impossible for integral \( t \) and \( u \).

Lemma 2. There are only finitely many \( m \) such that \( f_m \) is Rabinowitsch and \( 1 < D < m^{1/12} \).

Proof. Let \( p \) be the least prime with \( p \equiv 1 \pmod{4D} \) and \( (p,m) = 1 \). By Linnik’s theorem, we have \( p < D^C \) for some absolute constant \( C \), moreover, for \( D \) sufficiently large we may take \( C = 5.5 \), as shown by D. R. Heath-Brown [3]. Hence, there is some constant \( D_0 \) such that for \( D > D_0 \) we have \( p < m^{1/2}/6 \). By construction of \( p \), in any interval of length \( p \) there is some \( n \) such that \( x - \frac{1+\sqrt{D}}{2} \) is not coprime to \( p \), i.e. such that \( p \) divides \( n^2 + n - m \). If \( f_m \) is Rabinowitsch, this implies \( f_m(n) = \pm p \), since \( f_m \) is of degree 2, this cannot happen but for 4 values of \( n \). However, since \( p < m^{1/2}/6 \), in every interval of length \( m^{1/2} \), there are at least five such values of \( n \), hence, \( f_m \) is not Rabinowitsch.

Finally we choose a prime number \( p_D \equiv 1 \pmod{4D} \) for each \( D \leq D_0 \), and for \( m > 6 \max p_D \) we argue as above.

Lemma 3. There are only finitely many \( m \) such that \( f_m \) is Rabinowitch and that \( D \geq m^{1/12} \).

Proof. We may neglect the case \( m = 2 \). In each of the other cases, there exists a unit \( \epsilon_m \) in \( \mathbb{Q}(\sqrt{D}) \) with \( 1 < |\epsilon_m| \ll m \), more precisely, such a unit is given by
\[
\begin{align*}
m = t^2 & : \epsilon_m = 2t + \sqrt{4m+1} \\
m = t^2 + t + 1 & : \epsilon_m = \frac{2t+1 + \sqrt{4m+1}}{2} \\
m = t^2 + t \pm \frac{2t+1}{3} & : \epsilon_m = \frac{6t+3 \pm 2 + 3\sqrt{4m+1}}{2}
\end{align*}
\]
Let \( \epsilon_D > 1 \) be the fundamental unit of \( \mathbb{Q}(\sqrt{D}) \). Since the group of positive units in \( \mathbb{Q}(\sqrt{D}) \) is free abelian of rank 1, there is some \( k \) such that \( \epsilon_m = \epsilon_D^k \), hence we have \( \epsilon_D < m \). By the Siegel-Brauer-theorem we have \( \log(h(\mathbb{Q}(\sqrt{D}))) \log(\epsilon_D) \sim \log \sqrt{D} \). If \( f_m \) is Rabinowitch, then \( h(\mathbb{Q}(\sqrt{D})) = 1 \), and by assumption we have
\[
\log(\epsilon_D) \leq \log(\epsilon_m) < \log m \leq 12 \log D,
\]
which can only be true for finitely many \( D \). Since \( m \leq D^{12} \), there are only finitely many \( m \), and our claim follows.

Note that Lemma 1 and Lemma 2 are effective, while Lemma 3 depends on a bound for Siegel’s zero. However, one can deduce that there is an effective constant \( m_0 \), such that there exists at most one \( m > m_0 \) such that \( f_m \) is Rabinowitsch.
Note added in proof. In the mean time, D. Byeon and H. M. Stark [2] also obtained a proof of Theorem 1, moreover, they determined all Rabinowitsch polynomials up to at most one exception. The same result has also been obtained independently by S. Louboutin.

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