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(σ, τ)-DERIVATIONS ON PRIME NEAR RINGS

MOHAMMAD ASHRAF, ASMA ALI AND SHAKIR ALI

Abstract. There is an increasing body of evidence that prime near-rings with derivations have ring like behavior, indeed, there are several results (see for example [1], [2], [3], [4], [5] and [8]) asserting that the existence of a suitably-constrained derivation on a prime near-ring forces the near-ring to be a ring. It is our purpose to explore further this ring like behaviour. In this paper we generalize some of the results due to Bell and Mason [4] on near-rings admitting a special type of derivation namely (σ, τ)-derivation where σ, τ are automorphisms of the near-ring. Finally, it is shown that under appropriate additional hypothesis a near-ring must be a commutative ring.

1. Introduction

Throughout the paper N will denote a zero symmetric left near-ring with multiplicative centre Z. An element x of N is said to be distributive if (y + z)x = yx + zx for all x, y, z ∈ N. A near-ring N is called zero symmetric if 0x = 0 for all x ∈ N (recall that left distributivity yields x0 = 0). An additive mapping d : N → N is said to be a derivation on N if d(xy) = xd(y) + d(x)y for all x, y ∈ N or equivalently, as noted in [8], that d(xy) = d(x)y + x d(y) for all x, y ∈ N. Following [5], an additive mapping d : N → N is called a σ-derivation if there exists an automorphism σ : N → N such that d(xy) = σ(x)d(y) + d(x)y for all x, y ∈ N. Further this as a motivation we define an additive mapping d : N → N is called a (σ, τ)-derivation if there exists automorphisms σ, τ : N → N such that d(xy) = σ(x)d(y) + d(x)τ(y) for all x, y ∈ N. In case σ = 1, the identity mapping, d is called τ-derivation. Similarly if τ = 1, d is called σ-derivation. It is straightforward that an (1, 1)-derivation is ordinary derivation. For x, y ∈ N, the symbol [x, y] will denote the commutator xy − yx while the symbol (x, y) will denote the additive commutator x + y − x − y. Following [5] for x, y ∈ N, the symbol [x, y]σ,τ will denote the (σ, τ)-commutator σ(x)y − yτ(x) while (σ, τ)-derivation d will be called (σ, τ)-commuting if [x, d(x)]σ,τ = 0 for all x ∈ N. A near-ring N is

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said to be prime if \(aNb = (0)\) implies that \(a = 0\) or \(b = 0\). Further an element \(x \in N\) for which \(d(x) = 0\) is called a constant.

Some recent results on rings deal with commutativity of prime and semi-prime rings admitting suitably constrained derivations. It is natural to look for comparable results on near-rings and this has been done in [1], [2], [3], [4], [5] and [8]. It is our purpose to extend some of these results on prime near-rings admitting suitably constrained \((\sigma, \tau)\)-derivation.

2. Preliminary results

We begin with the following lemmas which are useful in sequel.

**Lemma 2.1.** An additive endomorphism \(d\) on a near-ring \(N\) is a \((\sigma, \tau)\)-derivation if and only if \(d(xy) = d(x) \tau(y) + \sigma(x) d(y)\), for all \(x, y \in N\).

**Proof.** Let \(d\) be a \((\sigma, \tau)\)-derivation on a near-ring \(N\). Since \(x(y + y) = xy + xy\), we obtain

\[
d(x(y + y)) = \sigma(x) d(y + y) + d(x) \tau(y + y)
\]

(2.1)

\[
= \sigma(x) d(y) + \sigma(x) d(y) + d(x) \tau(y) + d(x) \tau(y)
\]

for all \(x, y \in N\).

On the other hand, we have

\[
d(xy + xy) = d(xy) + d(xy)
\]

(2.2)

\[
= \sigma(x) d(y) + d(x) \tau(y) + \sigma(x) d(y) + d(x) \tau(y)
\]

for all \(x, y \in N\).

Combining (2.1) and (2.2), we find that

\[
\sigma(x) d(y) + d(x) \tau(y) = d(x) \tau(y) + \sigma(x) d(y), \quad \text{for all } x, y \in N.
\]

Thus, we have

(2.3)

\[
d(xy) = d(x) \tau(y) + \sigma(x) d(y), \quad \text{for all } x, y \in N.
\]

Conversely, let \(d(xy) = d(x) \tau(y) + \sigma(x) d(y)\), for all \(x, y \in N\). Then

\[
d(x(y + y)) = d(x) \tau(y + y) + \sigma(x) d(y + y)
\]

(2.4)

\[
= d(x) \tau(y) + d(x) \tau(y) + \sigma(x) d(y)
\]

\[
+ \sigma(x) d(y) \quad \text{for all } x, y \in N.
\]

Also,

\[
d(xy + xy) = d(xy) + d(xy)
\]

(2.5)

\[
= d(x) \tau(y) + \sigma(x) d(y) + d(x) \tau(y) + \sigma(x) d(y),
\]

for all \(x, y \in N\).

Combining (2.4) and (2.5), we obtain

\[
d(x) \tau(y) + \sigma(x) d(y) = \sigma(x) d(y) + d(x) \tau(y), \quad \text{for all } x, y \in N.
\]
Lemma 2.2. Let $d$ be a $(\sigma, \tau)$-derivation on the near-ring $N$. Then $N$ satisfies the following partial distributive laws:

(i) $(\sigma(x)d(y) + d(x)\tau(y))z = \sigma(x)d(y)z + d(x)\tau(y)z$, for all $x, y, z \in N$.

(ii) $(d(x)\tau(y) + \sigma(x)d(y))z = d(x)\tau(y)z + \sigma(x)d(y)z$, for all $x, y, z \in N$.

Proof. Note that for all $x, y, z \in N$,

(2.6) $d((xy)z) = \sigma(x)\sigma(y)d(z) + (\sigma(x)d(y) + d(x)\tau(y))\tau(z)$.

On the other hand, we have

\begin{equation}
(2.7) d(xyz) = \sigma(x)\sigma(y)d(z) + \sigma(x)d(y)\tau(z)
\end{equation}

+ $d(x)\tau(y)\tau(z)$, for all $x, y, z \in N$.

Equating (2.6) and (2.7), we find that

$(\sigma(x)d(y) + d(x)\tau(y))z = \sigma(x)d(y)z + d(x)\tau(y)z$, for all $x, y, z \in N$.

In the similar manner, (ii) can be proved. \qed

Lemma 2.3. Let $d$ be a $(\sigma, \tau)$-derivation on $N$ and suppose $u \in N$ is not a left zero divisor. If $[u, d(u)]_{\sigma, \tau} = 0$, then $(x, u)$ is a constant for every $x \in N$.

Proof. Since $u(u + x) = u^2 + ux$, so we obtain

$\sigma(u)d(x) + d(u)\tau(u) = d(u)\tau(u) + \sigma(u)d(x)$, for all $u \in N$ and $x \in N$.

Due to $[u, d(u)]_{\sigma, \tau} = 0$, the above expression can be written as

$\sigma(u)(d(x) + d(u)) = \sigma(u)(d(u) + d(x))$, for all $u, x \in N$

i.e.,

$\sigma(u)(d(x, u)) = 0$, for all $x \in N$.

Since $\sigma$ is an automorphism of $N$, $\sigma(u)$ is not a left-zero divisor. Thus $d(x, u) = 0$. Hence $(x, u)$ is constant, for all $x \in N$. \qed

Theorem 2.1. Let $N$ have no non-zero divisors of zero. If $N$ admits a non-trivial $(\sigma, \tau)$-commuting $(\sigma, \tau)$-derivation $d$, then $(N, +)$ is abelian.

Proof. Let $c$ be any additive commutator. Then application of Lemma 2.3 yields that $c$ is a constant. Moreover, for any $x \in N$, $xc$ is also an additive commutator, hence a constant. Thus, $0 = d(xc) = \sigma(x)d(c) + d(x)\tau(c)$ i.e. $d(x)\tau(c) = 0$, for all $x \in N$ and additive commutators $c$. Since $d(x) \neq 0$ for some $x \in N$, so $\tau(c) = 0$, and thus $c = 0$ for all additive commutators $c$. Hence, $(N, +)$ is abelian. \qed

3. Prime near-rings

Lemma 3.1. Let $N$ be a prime near-ring.

(i) If $z$ is a non-zero element in $Z$, then $z$ is not a zero divisor.

(ii) If there exists a non-zero element $z$ of $Z$ such that $z + z \in Z$, then $(N, +)$ is abelian.
(iii) Let \( d \) be a non-trivial \((\sigma, \tau)\)-derivation on \( N \). Then \( xd(N) = (0) \) or \( d(N)x = (0) \), implies \( x = 0 \).

(iv) If \( N \) is 2-torsion free and \( d \) is a \((\sigma, \tau)\)-derivation on \( N \) such that \( d^2 = 0 \) and \( \sigma, \tau \) commute with \( d \), then \( d = 0 \).

(v) If \( N \) admits a non-trivial \((\sigma, \tau)\)-derivation \( d \) for which \( d(N) \subseteq Z \), then \( c \in Z \) for each constant element \( c \) of \( N \).

**Proof.** (i) and (ii) are already proved in [4].

(iii) Let \( xd(r) = 0 \), for all \( r \in N \). Replace \( r \) by \( yz \), to get \( x\sigma(y) d(z) + x d(y) \tau(z) = 0 \), for all \( y, z \in N \). Hence we have \( x\sigma(y) d(z) = 0 \), for all \( y, z \in N \). Since \( \sigma \) is an automorphism of \( N \), \( xNd(N) = (0) \). Again \( N \) is prime and \( d(N) \neq 0 \), we have \( x = 0 \).

Arguing as above, we can show that \( d(r)x = 0 \), for all \( r \in N \), implies that \( x = 0 \).

(iv) For arbitrary \( x, y \in N \), we have \( d^2(xy) = 0 \). After a simple calculation, we obtain \( 2d(\sigma(x)) d(\tau(y)) = 0 \). Since \( N \) is 2-torsion free, so \( d(\sigma(x)) d(N) = (0) \), for each \( x \in N \). Hence \( d = 0 \), by using (iii) and the fact that \( \sigma \) is an automorphisms.

(v) Let \( c \) be an arbitrary constant and let \( x \) be a non-constant element of \( N \). Then \( d(x) \tau(c) = d(xc) \in Z \) for each non-constant element \( x \) of \( N \). This implies that \( d(x) \tau(c)y = y d(x) \tau(c) \), for all \( y \in N \). Since \( d(x) \in Z \setminus \{0\} \), it follows that \( d(x) \tau(c)y = d(x) y\tau(c) \), for all \( y \in N \) and we conclude that \( d(x)(yc - cy) = 0 \); for all \( y \in N \) and additive commutator \( c \). Hence, using (i), we get the required result. \( \square \)

**Theorem 3.1.** Let \( N \) be a prime near-ring admitting a non-trivial \((\sigma, \tau)\)-derivation \( d \) for which \( d(N) \subseteq Z \). Then \( (N, +) \) is abelian. Moreover, if \( N \) is 2-torsion free and \( \sigma, \tau \) commute with \( d \), then \( N \) is a commutative ring.

**Proof.** Since \( d(N) \subseteq Z \) and \( d \) is non-trivial, there exists a non-zero element \( x \) in \( N \) such that \( z = d(x) \in Z \setminus \{0\} \) and \( z + z = d(x + x) \in Z \). Hence \( (N, +) \) is abelian by Lemma 3.1(ii).

Assume now that, \( N \) is 2-torsion free and \( \sigma, \tau \) commute with \( d \). Application of Lemma 2.2 (i) yields that,

\[
(\sigma(x) d(y) + d(x) \tau(y))r = \sigma(x) d(y) r + d(x) \tau(y)r ,
\]

for all \( x, y, r \in N \).

Since \( d(N) \subseteq Z \), it follows that \( d(xy) \in Z \), for all \( x, y \in N \). Thus, \( d(xy) r = r d(xy) \), for all \( x, y, r \in N \) and hence

\[
(\sigma(x) d(y) + d(x) \tau(y))r = r(\sigma(x) d(y) + d(x) \tau(y))
\]

\[
= r\sigma(x) d(y) + r d(x) \tau(y) ,
\]

for all \( x, y, r \in N \).

Combine (3.1) and (3.2) and use the fact that \( (N, +) \) is abelian, to get

\[
\sigma(x) d(y) r - r\sigma(x) d(y) = r d(x) \tau(y) - d(x) \tau(y)r ,
\]

for all \( x, y, r \in N \).
Since $\sigma$ is an automorphism and $d(N) \subseteq Z$, the equation (3.3) can be rearranged to yield
\[ d(y)\sigma(x)r - r d(y)\sigma(x) = d(x)r\tau(y) - d(x)\tau(y)r, \quad \text{for all } x, y, r \in N \]
or
\[ d(y)(\sigma(x)r - r\sigma(x)) = d(x)(r\tau(y) - \tau(y)r), \quad \text{for all } x, y, r \in N. \tag{3.4} \]
Suppose on contrary that $N$ is not commutative and choose $r, y \in N$ with $r\tau(y) - \tau(y)r \neq 0$. Let $x = d(a)$, $a \in N$. This yields that $\sigma(x) = \sigma(d(a)) = d(\sigma(a)) \in Z$. Now (3.1) becomes $d(y)(d(\sigma(a))r - rd(\sigma(a)) = d^2(a)(r\tau(y) - \tau(y)r)$, i.e., $d^2(a)(r\tau(y) - \tau(y)r) = 0$, for all $a \in N$. By Lemma 3.1 (i), we see that the central element $d^2(a)$ can not be a divisor of zero, we conclude that $d^2(a) = 0$, for all $a \in N$. But by Lemma 3.1 (iv), this can not happen for non-trivial derivation $d$. Thus, $r\tau(y) - \tau(y)r = 0$, for all $r, y \in N$. Since $\tau$ is an automorphism of $N$, the above expression implies that $rz - zr = 0$, for all $r, z \in N$. Hence $N$ is a commutative ring.

**Theorem 3.2.** Let $N$ be a prime near-ring admitting a non-trivial $(\sigma, \tau)$-derivation $d$ such that $d(x)d(y) = d(y)d(x)$, for all $x, y \in N$. Then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion free and $\sigma, \tau$ commute with $d$, then $N$ is a commutative ring.

**Proof.** In view of our hypothesis, we have $d(x + x) d(x + y) = d(x + y) d(x + x)$, for all $x, y \in N$. This implies that $d(x) d(x) + d(x) d(y) = d(x) d(x) + d(y) d(x)$, for all $x, y \in N$ and hence $d(x) d(x, y) = 0$, for all $x, y \in N$ i.e., $d(x) d(c) = 0$, for all $x \in N$ and additive commutator $c$. Now, application of Lemma 3.1 (iii) yields that $d(c) = 0$, for all additive commutators $c$. Since $N$ is a left near-ring and $c$ is an additive commutator, $xc$ is also an additive commutator for any $x \in N$. Hence $d(xc) = 0$, for all $x \in N$ and additive commutator $c$. Thus by Lemma 3.1 (iii), $c = 0$ and hence $(N, +)$ is abelian. \(\square\)

Assume now that $N$ is 2-torsion free and $\sigma, \tau$ commute with $d$. Then applications of Lemmas 2.1 and 2.2 (i) yield that,
\[
d(d(x)y) d(z) = (d^2(x)\tau(y) + \sigma(d(x))d(y)) d(z) = d^2(x)\tau(y) d(z) + \sigma(d(x)) d(y) d(z) \quad \text{for all } x, y, z \in N.
\]
This implies that
\[ d^2(x)\tau(y) d(z) = d(d(x)y) d(z) - \sigma(d(x)) d(y) d(z), \quad \text{for all } x, y, z \in N. \tag{3.5} \]
Also, since $d(x) d(y) = d(y) d(x)$, for all $x, y \in N$, we find that
\[ d(d(xy))d(z) = d(z)d(d(xy)) \]
\[ = d(z)(d^2(x)\tau(y) + \sigma(d(x))d(y)) \]
\[ = d(z)d^2(x)\tau(y) + d(z)d(\sigma(x))d(y) \]
\[ = d^2(x)d(z)\tau(y) + \sigma(d(x))d(y)d(z) \]
for all \( x, y, z \in N \).

Combine (3.5) and (3.6), to get
\[ d^2(x)((\tau(y)d(z) - d(z)\tau(y)) = 0, \text{ for all } x, y, z \in N. \]

Now replacing \( y \) by \( yr \) in (3.7), we get
\[ d^2(x)(\tau(y)d(z) - d(z)\tau(y)) = 0, \text{ for all } r, x, y, z \in N. \]

Thus, \( d^2(x)N(\tau(r)d(z) - d(z)\tau(r)) = (0) \), for all \( r, x, z \in N \). Since \( N \) is prime and \( \tau \) is an automorphism, \( rd(z) - d(z)r = 0 \) or \( d^2(x) = 0 \), for all \( x \in N \). But the last conclusion is impossible by Lemma 3.1 (iv). Hence, we have \( rd(z) - d(z)r = 0 \), for all \( r, z \in N \). This implies that \( d(N) \subseteq Z \). Hence \( N \) is a commutative ring by Theorem 3.1.

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