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these Terms of use.
ON TOTALLY REAL MINIMAL SUBMANIFOLDS
IN COMPLEX PROJECTIVE SPACE

XIAOLI CHAO AND YAOWEN LI

ABSTRACT. In this paper, we obtain some pinching theorems for totally real minimal submanifolds in complex projective space.

§1. INTRODUCTION

Let $CP^n(c)$ be an $n$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $c (c > 0)$. The pinching problem for totally real minimal submanifolds in $CP^n(c)$ has been studied by many mathematicians. Montiel, Ros and Urbano [MRU] proved a pinching result about Ricci curvature condition. Recently, Matsuyama [M1,2] has discussed the scalar curvature case which give a positive answer for Ogiue’s conjecture [O]. Now, in this paper, we give a pinching condition for the norm of the second fundamental form under which the submanifolds is totally geodesic.

Throughout this paper, we use the similar notations and formulas as those used in [MRU]. Let $M$ be an n-dimensional compact Riemannian manifold. We denote by $UM$ the unit tangent bundle over $M$ and by $UM_p$ its fibre at $p \in M$. For any continuous function $f: UM \to R$, we have

$$\int_{UM} f dv = \int_M \int_{UM_p} f dv_p dp$$

where $dp$, $dv_p$ and $dv$ stand for the canonical measures on $M$, $UM_p$ and $UM$ respectively.

If $T$ is a k-covariant tensor on $M$ and $\nabla T$ is covariant derivative, then we have ($[R1]$)

$$\int_{UM} \left\{ \sum_{i=1}^n (\nabla T)(e_i, e_i, v, \cdots, v) \right\} dv = 0$$

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where \( e_1, \ldots, e_n \) is an orthonormal basis of \( T_p M, p \in M \).

Suppose now that \( M \) is isometrically immersed in an \((n + p)\)-dimensional Riemannian manifold \( \overline{M}^{n+p} \). We denote by \( \langle , \rangle \) the metric of \( \overline{M} \) as well as that induced on \( M \). Let \( \sigma \) be the second fundamental form of the isometrically immersion and \( A_\xi \) the Weingarten endomorphism for a normal vector \( \xi \). If \( T_p M \) and \( T_p^\perp M \) denote the tangent and normal spaces to \( M \) at \( p \), one can define

\[
L: T_p M \rightarrow T_p M \quad \text{and} \quad T: T_p^\perp M \times T_p^\perp M \rightarrow R
\]

by the expressions

\[
Lv = \sum_{i=1}^{n} A_{\sigma(v, e_i)} e_i \quad \text{and} \quad T(\xi, \eta) = \text{trace} A_\xi A_\eta
\]

where \( e_1, \ldots, e_n \) is an orthonormal basis of \( T_p M \). Then \( L \) is a self-adjoint linear map and \( T \) a symmetric bilinear map.

There are many submanifolds satisfying \( T = k\langle , \rangle \). Obviously, hypersurfaces represent a trivial case. In \( CP^{n+p}(c) \), a Kaehler submanifold of order \( \{k_1, k_2\} \) for some natural numbers \( k_1 \) and \( k_2 \) is one submanifold of this type ([R3]). In this paper, we have a pinching theorem for this kind of submanifolds as following:

**Theorem 3.1.** Let \( M^n \) be a totally real minimal submanifold with \( T = k\langle , \rangle \) in \( CP^{n+p}(c) \). If

\[
|\sigma|^2 < \frac{n c (n + 2p)(n + 4)}{4(n + 2)(n + 4) + n(n + 4)^2 + 4n},
\]

then \( M \) must be totally geodesic.

\[\text{Remark.} \quad \text{When the immersion is minimal, Lemma 2.1 is due to [MRU].}\]

\[\text{Remark.} \quad \text{It's clear that submanifolds in real space forms, Kahler, and totally real submanifolds in complex space forms are curvature-invariant.}\]
Lemma 2.2. Let $M$ be an $n$-dimensional compact submanifold isometrically immersed in a Riemannian manifold $\mathbb{M}^{n+p}$. Then, for $\forall p \in M$, we have:

i) $\int_{U M_p} \langle L v, A_{\sigma(v,v)} v \rangle \, dv_p = \frac{2}{n+2} \int_{U M_p} |L v|^2 \, dv_p + \frac{1}{n+2} \int_{U M_p} \langle \sigma(v,v), \xi \rangle \, dv_p$

ii) $\int_{U M_p} |\sigma(v,v)|^2 \, dv_p = \frac{2}{n+2} \int_{U M_p} \langle L v, v \rangle \, dv_p + \frac{1}{n+2} \int_{U M_p} \sum_{i=1}^{n} \langle \sigma(v,v), \sigma(e_i,e_i) \rangle \, dv_p$

iii) $\int_{U M_p} \langle L v, v \rangle \, dv_p = \frac{1}{n} \int_{U M_p} |\sigma|^2 \, dv_p$

iv) $\int_{U M_p} \langle \sigma(v,v), \eta \rangle \, dv_p = \frac{1}{n} \int_{U M_p} \sum_{i=1}^{n} \langle \sigma(e_i,e_i), \eta \rangle \, dv_p$

Where $\xi = \sum_{i=1}^{n} \sigma(e_i,Le_i)$ and $\eta$ is a fixed vector in normal bundle.

Proof. Let $\alpha^1$ be the 1-form on $U M_p$ defined by

$\alpha^1(e) = \langle L v, A_{\sigma(v,v)} e \rangle, \quad v \in U M_p, \quad e \in T_v U M_p$

For any $v \in U M_p$, let $e_1, \ldots, e_{n-1}, e_n = v$ be an orthonormal basis of $T_p M$. Then

$$(\delta \alpha^1)(v) = -(n+2) \langle L v, A_{\sigma(v,v)} v \rangle + 2 |L v|^2 + \langle \sigma(v,v), \xi \rangle.$$ Integrating it over $U M_p$, we obtain i).

ii), iii) and iv) are obtained by using the same technique for the 1-forms $\alpha^2$, $\alpha^3$ and $\alpha^4$ on $U M_p$ defined by

$\alpha^2(e) = \langle \sigma(v,v), \sigma(v,e) \rangle$

$\alpha^3(e) = \langle L v, e \rangle$

$\alpha^4(e) = \langle \sigma(v,e), \eta \rangle$

□

Lemma 2.3. Let $M$ be an $n$-dimensional compact submanifold isometrically immersed in a Riemannian manifold $\mathbb{M}^{n+p}$. Then we have

$$\int_{U M_p} \left| A_{\sigma(v,v)} v \right|^2 \, dv_p \geq \frac{2}{n+2} \int_{U M_p} \langle L v, A_{\sigma(v,v)} v \rangle \, dv_p$$

$$+ \frac{1}{n+2} \int_{U M_p} \langle A_{\sigma(e_i,e_i)} v, A_{\sigma(v,v)} v \rangle \, dv_p$$
**Proof.** Let \( \triangle \) denote the Laplace operator on \( S^{n-1} \). Then, for the function \( f : UM_p \to T_p M \) defined by \( f(v) = A_{\sigma(v,v)}v \), we have

\[
(\triangle f)(v) = -3(n+1)A_{\sigma(v,v)}v + 4Lv + 2A_{\sigma(e_i,e_i)}v.
\]

Since \( UM_p \) is a \((n-1)\)-dimensional sphere, the first eigenvalue of \(-\triangle = \nabla \nabla_{e_i} e_i - \nabla_{e_i} \nabla_{e_i}\) is \( n-1 \). Then

\[
-\int_{UM_p} \langle \triangle f, f \rangle dv_p \geq (n-1) \int_{UM_p} |f|^2 dv_p
\]

and the lemma follows. \( \square \)

Let \( \alpha \) be a 1-form on \( UM_p \) defined by

\[
\alpha_v(e) = \langle A_{\sigma(v,e)}e, A_{\sigma(v,v)}v \rangle
\]

where \( v \in UM_p \), and \( e \in T_v UM_p \). If \( e_1, \ldots, e_{n-1} \) is an orthonormal basis of \( T_v UM_p \), then the codifferential of \( \alpha \) is

\[
(\delta \alpha)(v) = \sum_{i=1}^{n} e_i \cdot \alpha_v(e_i)
\]

\[
= -(n+4)|A_{\sigma(v,v)}v|^2 + 2\langle Lv, A_{\sigma(v,v)}v \rangle
\]

\[
+ T(\sigma(v,v), \sigma(v,v)) + 2 \sum_{i=1}^{n} \langle A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}v \rangle,
\]

where \( e_1, \ldots, e_{n-1}, e_n = v \) is an orthonormal basis of \( T_p M \). Now integrating the above equality over \( UM_p \) and using divergence theorem, we have

\[
2 \int_{UM_p} \left\{ \sum_{i=1}^{n} \langle A_{\sigma(v,v)}e_i, A_{\sigma(v,v)}v \rangle \right\} dv_p
\]

\[
= (n+4) \int_{UM_p} |A_{\sigma(v,v)}v|^2 dv_p - 2 \int_{UM_p} \langle Lv, A_{\sigma(v,v)}v \rangle dv_p
\]

\[
- \int_{UM_p} T(\sigma(v,v), \sigma(v,v)) dv_p
\]

(2.1)

In a similar way, for the 1-form \( \alpha \) defined by

\[
\alpha_v(e) = \langle A_{\sigma(v,e)}v, A_{\sigma(v,v)}v \rangle,
\]

we have

\[
(\delta \alpha)(v) = \sum_{i=1}^{n} \left\{ 2|A_{\sigma(v,e_i)}v|^2 + \langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i \rangle
\]

\[
+ \langle A_{\sigma(e_i,e_i)}v, A_{\sigma(v,v)}v \rangle \right\} - (n+4)|f(v)|^2 + \langle Lv, f(v) \rangle.
\]
Integrating this and using (2.1), we get

\[ 2 \int_{U M_p} \sum_{i=1}^{n} |A_{\sigma(v,e_i)}v|^2 dv_p = \int_{U M_p} \left\{ \frac{n+4}{2} |f(v)|^2 - \langle A_{nH}v, f(v) \rangle \right\} dv_p + \frac{1}{2} T(\sigma(v,v), \sigma(v,v)) \]

(2.2)

By (2.1), (2.2) and

\[ 2 \sum_{i=1}^{n} \langle A_{\sigma(v,e_i)}v, A_{\sigma(v,v)}e_i \rangle \leq a \sum_{i=1}^{n} |A_{\sigma(v,e_i)}v|^2 + \frac{1}{a} \sum_{i=1}^{n} |A_{\sigma(v,v)}e_i|^2 \]

(2.3)

\[ = a \sum_{i=1}^{n} |A_{\sigma(v,e_i)}v|^2 \frac{1}{a} T(\sigma(v,v), \sigma(v,v)) \]

By (2.1), (2.2) and (2.3), we have, for \( \forall b > 0 \),

\[ \int_{U M_p} \left\{ \left( n + 4 - \frac{b(n+4)}{4} \right) |f(v)|^2 - 2\langle Lv, f(v) \rangle - \left( 1 + \frac{b}{4} + \frac{1}{b} \right) T(\sigma(v,v), \sigma(v,v)) \right\} dv \leq 0. \]

(2.4)

Now, we can prove the following lemma:

**Lemma 2.4.** Let \( M^n \to \overline{M}^{n+p} \) be a compact Riemannian immersion. Then we have

\[ (1) \quad \int_{U M_p} (n + 2) \langle A_H v, f(v) \rangle dv_p = \int_{U M_p} \left\{ 2 \sum_{i=1}^{n} \langle A_{He_i}, A_{\sigma(v,e_i)}v \rangle + T(H, \sigma(v,v)) \right\} dv_p \]

(2)

\[ \int_{U M_p} \langle A_H v, Lv \rangle dv_p = \int_{U M_p} \sum_{i=1}^{n} \langle A_{He_i}, A_{\sigma(v,e_i)}v \rangle dv_p \]

(3)

\[ \int_{U M_p} \langle A_H v, Lv \rangle dv_p = \frac{1}{n} \int_{U M_p} \sum_{i=1}^{n} \langle A_{He_i}, L e_i \rangle dv_p \]

\[ = \frac{1}{n} \int_{U M_p} \langle H \cdot \xi \rangle dv_p \]

(4)

\[ \int_{U M_p} T(H < \sigma(v,v)) dv_p = \int_{U M_p} T(H, H) dv_p \]
\[ \int_{UM_p} (n + 2)T(\sigma(v, v), \sigma(v, v)) \, dv_p \]

\[ = \int_{UM_p} \left\{ nT(H, \sigma(v, v)) + 2 \sum_{i=1}^{n} T(\sigma(v, e_i), \sigma(v, e_i)) \right\} \, dv_p \]

\[ \int_{UM_p} \sum_{i=1}^{n} T(\sigma(v, e_i), \sigma(v, e_i)) \, dv_p = \frac{1}{n} \int_{UM_p} \sum_{i,j=1}^{n} T(\sigma(e_i, e_j), \sigma(e_i, e_j)) \, dv_p \]

\[ \int_{UM_p} \left\langle A_H v, f(v) \right\rangle \, dv_p = \int_{UM_p} \left\{ \frac{1}{n + 2} T(H, H) + \frac{2}{n(n + 2)} (H, \xi) \right\} \, dv_p \]

\[ \int_{UM_p} T(\sigma(v, v), \sigma(v, v)) \, dv_p = \int_{UM_p} \left\{ \frac{n}{n + 2} T(H, H) + \frac{2}{n(n + 2)} \sum_{i,j=1}^{n} T(\sigma(e_i, e_j), \sigma(e_i, e_j)) \right\} \, dv_p \]

\[ \int_{UM_p} \left( 2 - \frac{b(n + 4)}{4} \right) |f(v)|^2 \, dv_p \]

\[ \leq \int_{UM_p} \left\{ \left( 1 + \frac{b}{4} + \frac{1}{b} \right) T(\sigma(v, v), \sigma(v, v)) - \left( 1 + \frac{b}{2} \right) n \left\langle A_H v, f(v) \right\rangle \right\} \, dv_p , \]

for each \( b \).

**Proof.** By taking some proper 1-form on \( UM_p \) respectively as above, we can obtain (1) \( \sim \) (6) and then (7) and (8) as their corollaries. Using Lemma 2.3, (2.4) implies (9). \( \square \)

**Remark.** When \( b(> 0) \) is small, (9) gives a estimation of the upper bound of \( |f(v)|^2 \).

### §3. Totally real submanifolds with \( T = k\langle , \rangle \) in complex projective spaces

There are many submanifolds satisfying \( T = k\langle , \rangle \). Obviously, hypersurfaces represent a trivial case. In \( CP^{n+p}(c) \), a Kaehler submanifold of order \( \{ k_1, k_2 \} \) for
some natural numbers \(k_1\) and \(k_2\) is one submanifold of this type ([R3]). Let \(M^n\) be a totally real minimal submanifold with \(T = k\langle,\rangle\) immersed in \(CP^{n+p}(c)\). Then

\[
P(R) = \sum_{i=1}^{n} R(e_i, v, \sigma(v, e_i), \sigma(v, v)) + 2 \sum_{i=1}^{n} R(e_i, v, v, A_{\sigma(v, e_i)}v)
= \frac{c}{2} \langle Lv, v \rangle - \frac{c}{2} |\sigma(v, v)|^2 + \frac{c}{4} \sum_{i=1}^{n} \langle \sigma(v, v), Je_i \rangle^2
- \frac{c}{4} \sum_{i=1}^{n} \langle Jv, \sigma(e_i, e_i) \rangle \langle Jv, \sigma(v, v) \rangle.
\]

(3.1)

Now, we define a map \(g^1 : UM_p \rightarrow T_p M\) by

\[
g^1(v) = A_{\sigma(v, v)}v - Lv.
\]

By a direct computation, we have

\[
(-\triangle g^1)(v) = 3(n + 1)f(v) - (n + 3)Lv - 2nA_Hv.
\]

Here \(\triangle\) is the Laplacian of \(UM_p\). Since \(\int_{UM_p} g^1(v) dv_p = 0\), we get

\[
\int_{UM_p} \langle (-\triangle g^1)(v), g^1(v) \rangle \geq (n - 1) \int_{UM_p} |g^1(v)|^2.
\]

Then, the above relation gives

\[
\int_{UM_p} \left\{(2n + 4)|f(v)|^2 - (2n + 8)\langle Lv, f(v) \rangle - 2n\langle f(v), A_H v \rangle + 4|Lv|^2 + 2n\langle Lv, A_H v \rangle \right\} dv_p \geq 0.
\]

(3.2)

In a similar way, for the 1-form \(g^2(v) = f(v) + Lv\), we have

\[
\int_{UM_p} \left\{(2n + 4)|f(v)|^2 - 2n\langle Lv, f(v) \rangle - 2n\langle f(v), A_H v \rangle - 4|Lv|^2 - 2n\langle Lv, A_H v \rangle \right\} dv_p \geq 0.
\]

(3.3)

By (3.2) and (3.3), we get

\[
\int_{UM_p} \left\{(2n + 4)|f(v)|^2 - (2kn + 4k + 4)\langle Lv, f(v) \rangle - 2n\langle f(v), A_H v \rangle + 4k|Lv|^2 - 2nk\langle Lv, A_H v \rangle \right\} dv_p \geq 0.
\]

(3.4)
Since $M$ is minimal, by (3.4) with $k = -\frac{2}{n+2}$, we have
\[
\int_{UM_p} |f(v)|^2 dv_p \geq \frac{4}{(n+2)^2} \int_{UM_p} |L v|^2 dv_p.
\]
From this and Lemma 2.2 i) we get
\[
\int_{UM_p} |f(v)|^2 dv_p \geq \frac{2}{n+2} \int_{UM_p} \langle L v, f(v) \rangle dv_p.
\]
From (3.1), (3.5) and Lemma 2.1 we have
\[
0 = \int_{UM} \left\{ \sum_{i=1}^n |(\nabla \sigma)(e_i, v, v)|^2 + (n+4)|f(v)|^2 \\
- 4\langle L v, f(v) \rangle - 2T(\sigma(v, v), \sigma(v, v)) \\
+ \left[ \frac{c}{2} \langle L v, v \rangle - \frac{c}{2} |\sigma(v, v)|^2 + \frac{c}{4} \sum_{i=1}^n \langle \sigma(v, v), J e_i \rangle \right]^2 \right\} dv
\geq \int_{UM} \left\{ \sum_{i=1}^n |(\nabla \sigma)(e_i, v, v)|^2 + \frac{nc}{4} |\sigma(v, v)|^2 \\
- n|f(v)|^2 - 2T(\sigma(v, v), \sigma(v, v)) \right\} dv.
\]
Assuming now that $M$ is minimal, and putting $b = \frac{4}{n+4}$ in formula (9) of Lemma 2.4, we obtain
\[
\int_{UM_p} |f(v)|^2 dv_p \leq \left( 1 + \frac{1}{n+4} + \frac{n+4}{4} \right) \int_{UM_p} T(\sigma(v, v), \sigma(v, v)) dv_p.
\]
By (3.6), (3.7) and the fact that $T = \frac{|\sigma|^2}{2p+n}g$ we get
\[
0 \geq \int_{UM} \left\{ \sum_{i=1}^n |(\nabla \sigma)(e_i, v, v)|^2 \\
+ \left[ \frac{nc}{4} - \frac{n(1 + \frac{1}{n+4} + \frac{n+4}{4})}{2p+n} |\sigma|^2 \right] \cdot |\sigma(v, v)|^2 \right\} dv.
\]
From this we immediately have

**Theorem 3.1.** Let $M^n$ be a totally real minimal submanifold with $T = k\langle , \rangle$ in $CP^{n+p}(c)$. If
\[
|\sigma|^2 < \frac{nc(n+2p)(n+4)}{4(n+2)(n+4) + n(n+4)^2 + 4n},
\]
then $M$ must be totally geodesic.

**Remark.** Xia [X] gave a pinching constant $\frac{n+6}{6}$ without the assumption: $T = k\langle \cdot, \cdot \rangle$. When $p > \frac{n(n+4)}{12} + \frac{2}{3} + \frac{n}{3(n+4)} - \frac{n}{6}$, our pinching constant is larger than Xia’s.

**Remark.** When the target manifold is the quaternionic space form $QP^{n+p}(c)$, we have also a corresponding result, i.e., changing the factor $n + 2p$ in (3.8) to $3n + 4p$. So our result is better than that of [Sh1] in case when $p$ is large enough.

**Remark.** B. Y. Chen and K. Ogiue ([CO]) had proved that, for a submanifold $M$ of nonflat complex space form, $M$ is curvature-invariant if and only if $M$ is holomorphic or totally real submanifold. So we can use Lemma 2.1 in the proof of Theorem 3.1.

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