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BOUNDARY VALUE PROBLEMS FOR FIRST ORDER MULTIVALUED DIFFERENTIAL SYSTEMS

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Abstract. We present some existence results for boundary value problems for first order multivalued differential systems. Our approach is based on topological transversality arguments, fixed point theorems and differential inequalities.

1. Introduction

In this paper we investigate boundary value problems for first order multivalued differential systems. More specifically, we shall be concerned with the existence of solutions of the following boundary value problem for first order differential inclusions

$$x'(t) \in A(t)x(t) + F(t, x(t)) \;, \quad t \in (0,1); \quad Mx(0) + Nx(1) = 0 \tag{1}$$

Here $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a Carathéodory multifunction, $I = [0,1]$, $A(.)$ is a continuous $n \times n$ matrix function, M and N are constant $n \times n$ matricies. We shall denote by $\|x\|$ the norm of any element $x \in \mathbb{R}^n$ and by $\|A\|$ the norm of any matrix A. Several authors have investigated problems similar to (1) under various assumptions (see for instance [1], [2], [3], [4], [5], [7], [10], [11], [12], [16] and the references therein). Problems (1) appear in the description of many physical phenomena; for example dry friction problems (see for instance [9] and [19]), control problems (see [8], [13], [16] and the references therein). We shall present existence results under fairly general conditions on the multifunction F, the matrices A, M and N. Our approach is based on the topological transversality theorem due to Granas, fixed point theorems and differential inequalities. For the use of the topological degree in multivalued boundary value problems we refer the reader to [18]. Our results are based on different assumptions than those published earlier, and cannot be derived trivially from the above cited results.

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2. Preliminaries

In this section we introduce notations, definitions and results that will be used in the remainder of the paper.

2.1. **Set-valued maps.** Let X and Y be Banach spaces. A set-valued map $G : X \to 2^Y$ is said to be compact if $G(X) = \overline{\cup\{G(x); x \in X\}}$ is compact. G has convex (closed, compact) values if $G(x)$ is convex (closed, compact) for every $x \in X$. G is bounded on bounded subsets of X if $G(B)$ is bounded in Y for every bounded subset B of X. A set-valued map G is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set O containing Gz_0, there exists a neighborhood $\mathcal{M}$ of z_0 such that $G(\mathcal{M}) \subset O$. G is usc on X if it is usc at every point of X. If G is nonempty and compact-valued then G is usc if and only if G has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $bcc(X)$. A set-valued map $G : I \to bcc(X)$ is measurable if for each $x \in X$, the function $t \mapsto \operatorname{dist}(x, G(t))$ is measurable on I. If $X \subset Y$, G has a fixed point if there exists $x \in X$ such that $x \in Gx$. Also, $|G(x)| = \sup\{|y|; y \in G(x)\}$.

Definition 1. A multivalued map $F : I \times \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ is said to be an L^1--Carathéodory multifunction if

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}^n$;
(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in I$;
(iii) For each $\sigma > 0$, there exists $h_\sigma \in L^1(I, \mathbb{R}_+)$ such that

$$\|x\| \leq \sigma \Longrightarrow \|F(t,x)\| = \sup\{\|v\| : v \in F(t,x)\} \leq h_\sigma(t) \text{ a.e. } t \in I\,.$$

$S^1_{F(.,x(.))} = \{v \in L^1(I, \mathbb{R}^n) : v(t) \in F(t, x(t)) \text{ for a.e. } t \in I\}$ denotes the set of selectors of F that belong to L^1. By a solution of (1) we mean an absolutely continuous function x on I, such that

$$x'(t) = A(t)x(t) + f(t)\,, \quad \text{a.e.} \quad t \in (0,1);\ Mx(0) + Nx(1) = 0 \tag{2}$$

where $f \in S^1_{F(.,x(.))}$. $AC_0(I)$ denotes the space of absolutely continuous functions x on I with $Mx(0) + Nx(1) = 0$. Also, for $x \in AC(I)$ we define its norm by $\|x\|_0 = \sup\{\|x(t)\|\,; t \in I\}$.

Note that for an L^1-Carathéodory multifunction $F : I \times \mathbb{R}^n \longrightarrow 2^{\mathbb{R}^n}$ the set $S^1_{F(.,x(.))}$ is not empty (see [14]).

For more details on set-valued maps we refer to [6] and [8].

2.2. **Topological transversality theory for set-valued maps.** (see [11]).

Let X be a Banach space, C a convex subset of X and U an open subset of C. $K_{\partial U}(\overline{U}, 2^C)$ shall denote the set of all set-valued maps $G : \overline{U} \to 2^C$ which are compact, usc with closed convex values and have no fixed points on ∂U (i.e., $u \notin Gu$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0,1] \times \overline{U} \to 2^C$ which is compact, usc with closed convex values. If $u \notin H(\lambda, u)$ for every $\lambda \in [0,1], u \in \partial U$, H is said to be fixed point free on ∂U. Two set-valued maps $F, G \in K_{\partial U}(\overline{U}, 2^C)$ are called homotopic in $K_{\partial U}(\overline{U}, 2^C)$ if there exists a compact homotopy $H : [0,1] \times \overline{U} \to 2^C$ which is fixed point free on ∂U and such that $H(0,\cdot) = F$ and $H(1,\cdot) = G$. $G \in K_{\partial U}(\overline{U}, 2^C)$ is called essential if every $F \in K_{\partial U}(\overline{U}, 2^C)$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise G is called inessential.

Theorem 1 (Topological transversality theorem)**.** *Let F, G be two homotopic set-valued maps in $K_{\partial U}(\overline{U}, 2^C)$. Then F is essential if and only if G is essential.*

Theorem 2. *Let $G : \overline{U} \to 2^C$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, G is essential.*

Theorem 3 (Theorem 2.1 in [17])**.** *Let U be an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$, $G(\overline{U})$ is bounded and $G : \overline{U} \to C$ is given by $G = G_1 + G_2$ where $G_1 : \overline{U} \to E$ is continuous and completely continuous, and $G_2 : \overline{U} \to E$ is a nonlinear contraction (i.e. there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|G_2(x) - G_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \overline{U}$). Then either,*

(A1) *G has a fixed point in $\overline{U}$; or*

(A2) *there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda G(u)$.*

Remark 1. This theorem is stated in terms of single-valued maps. However, it follows from the proof given in [17] that the theorem is still valid if G_1 is a multivalued operator. Also, we shall apply this theorem with $G_2 \equiv 0$, the identically zero single-valued map.

3. Main results

In this section, we state and prove our main results.

3.1. **A linear problem.** Consider the following linear boundary value problem

$$x'(t) = A(t)x(t) + h(t), \quad \text{a.e.} \quad t \in (0,1); \quad Mx(0) + Nx(1) = 0. \tag{3}$$

Let $\Phi(t)$ be a fundamental matrix solution of $x'(t) = A(t)x(t)$, such that $\Phi(0) = I$, the $n \times n$ identity matrix. Then any solution $x'(t) = A(t)x(t)$ is given by $x(t) = \Phi(t)v$ where v is an arbitrary constant vector. The boundary condition $Mx(0) + Nx(1) = 0$ implies that $M\Phi(0)v + N\Phi(1)v = 0$ or equivalently $(M + N\Phi(1))v = 0$. It follows that the homogeneous problem $x'(t) = A(t)x(t)$, $Mx(0) + Nx(1) = 0$ has only the trivial solution if and only if $\det(M + N\Phi(1)) \neq 0$. In this case the linear nonhomogeneous problem (3) has a unique solution given by $x(t) = \int_0^1 G(t,s)h(s)\,ds$ where $G(t,s)$ is the Green's matrix. Simple computations give

$$G(t,s) = \begin{cases} \Phi(t)\,J(s) & 0 \le t \le s \\ \Phi(t)\,\Phi(s)^{-1} + \Phi(t)\,J(s) & s \le t \le 1 \end{cases}$$

where $J(t) = -(M + N\Phi(1))^{-1}N\Phi(1)\Phi(t)^{-1}$.

Let $G_0 := \sup\{\|G(t,s)\|\,; (t,s) \in I \times I\}$.

We shall assume throughout the paper that $A(.)$ is a continuous matrix function on I with $A_0 := \sup\{\|A(t)\|\,; t \in I\}$, and the matrices M and N satisfy $\det(M + N\Phi(1)) \neq 0$.

Our first result is based on the following assumption.

(H1) $F : I \times \mathbb{R}^n \to bcc(\mathbb{R}^n)$ is an L^1-Carathéodory multifunction satisfying

$$\|F(t,x)\| \leq \alpha(t)\,\psi(\|x\|) \quad \text{for a.e. } t \in I, \text{ all } x \in \mathbb{R}^n,$$

where $\alpha \in L^1 I; \mathbb{R}_+$ and $\psi : [0,+\infty) \to (0,+\infty)$ is continuous nondecreasing and such that

$$\limsup_{\rho \to +\infty} \frac{\rho}{\psi(\rho)} = +\infty .$$

Our first result reads as follows.

Theorem 4. *If the assumption* (H1) *is satisfied, then the boundary value problem* (1) *has at least one solution.*

Proof. This proof will be given in several steps.

Step 1. Consider the set-valued operator $\mathcal{F} : AC(I) \to L^1(I)$ defined by

$$(\mathcal{F}x)(t) = F(t, x(t)) .$$

$\mathcal{F}$ is well defined, usc, with convex values and sends bounded subsets of $AC(I)$ into bounded subsets of $L^1(I)$. In fact, we have

$$\mathcal{F}x := \{u : I \to \mathbb{R}^n \text{ measurable}; \, u(t) \in F(t,x(t)) \text{ a.e. } t \in I\} .$$

Let $z \in AC(I)$. If $u \in \mathcal{F}z$ then

$$\|u(t)\| \leq \alpha(t)\,\psi(\|z(t)\|) \leq \alpha(t)\,\psi(\|z\|_0) .$$

Hence $\|u\|_{L^1} \leq C_0 := \|\alpha\|_{L^1}\,\psi(\|z\|_0)$. This shows that $\mathcal{F}$ is well defined. It is clear that $\mathcal{F}$ is convex valued.

Now, let B be a bounded subset of $AC(I)$. Then, there exists $K > 0$ such that $\|u\|_0 \leq K$ for $u \in B$. So, for $w \in \mathcal{F}u$ we have $\|w\|_{L^1} \leq C_1$, where $C_1 = \psi(K)\,\|\alpha\|_{L^1}$.

Also, we can argue as in [10, p. 16] to show that $\mathcal{F}$ is usc.

Step 2. A priori bounds on solutions.

Let x be a possible solution of (1). Then there exists a positive constant R^*, independent of x, such that

$$|x(t)| \leq R^* \quad \text{for all} \quad t \in I .$$

For, it follows from the definition of solutions of (1) that

$$x'(t) = A(t)x(t) + f(t) \quad \text{a.e.} \quad t \in (0,1)\,; Mx(0) + Nx(1) = 0$$

where $f \in S^1_{F(.,x(.))}$. It is clear that the solution of the above problem is given by

$$x(t) = \int_0^1 G(t,s) f(s)\, ds . \tag{4}$$

Hence

$$\|x(t)\| \leq \int_0^1 \|G(t,s)\|\, \|f(s)\|\, ds . \tag{5}$$

Assumption (H1) yields

$$\|x(t)\| \leq G_0 \int_0^1 \alpha(s)\,\psi(\|x(s)\|)\,ds\,. \tag{6}$$

Let

$$R_0 = \max\{\|x(t)\|\,; t \in J\}\,.$$

Then

$$R_0 \leq G_0 \int_0^1 \alpha(s)\,\psi(\|x(s)\|)\,ds\,. \tag{7}$$

Since ψ is nondecreasing we have

$$R_0 \leq G_0 \int_0^1 \alpha(s)\,\psi(R_0)\,ds\,. \tag{8}$$

The last inequality implies that

$$\frac{R_0}{\psi(R_0)} \leq G_0\,\|\alpha\|_{L^1}\,. \tag{9}$$

Now, the condition on ψ in (H1) shows that there exists $R^* > 0$ such that for all $R > R^*$

$$\frac{R}{\psi(R)} > G_0\,\|\alpha\|_{L^1}\,. \tag{10}$$

Comparing these last two inequalities (9) and (10) we see that $R_0 \leq R^*$. Consequently, we obtain $\|x(t)\| \leq R^*$ for all $t \in I$.

Step 3. Existence of solutions.

For $0 \leq \lambda \leq 1$ consider the one-parameter family of problems

$$x'(t) \in A(t)x(t) + \lambda F(t, x(t)) \quad t \in I,\ Mx(0) + Nx(1) = 0 \tag{1_λ}$$

which reduces to (1) for $\lambda = 1$.

It follows from Step 2 that if x is a solution of $(1)_\lambda$ for some $\lambda \in [0,1]$, then

$$\|x(t)\| \leq R^* \quad \text{for all} \quad t \in I$$

and R^* does not depend on λ.

Define $\mathcal{F}_\lambda : C(I) \to L^1(I)$ by

$$(\mathcal{F}_\lambda x)(t) = \lambda F(t, x(t))\,.$$

Step 1 shows that $\mathcal{F}_\lambda$ is usc, has convex values and sends bounded subsets of $AC(I)$ into bounded subsets of $L^1(I)$. Let $j : AC_0(I) \to AC(I)$ be the continuous embedding. The operator $L : AC_0(I) \to L^1(I)$, defined by $(Lx)(t) = x'(t) - A(t)x(t)$ has a bounded inverse (in fact this follows from the solution given by (4)), which we denote by L^{-1}. Moreover L^{-1} is completely continuous.

Let $B_{R^*+1} := \{x \in AC_0(I); \|x\|_0 < R^* + 1\}$. Define a set-valued map $H : [0,1] \times B_{R^*+1} \to AC_0(I)$ by

$$H(\lambda, x) = (L^{-1} \circ \mathcal{F}_\lambda \circ j)(x)\,.$$

We can easily show that the fixed points of $H(\lambda,\cdot)$ are solutions of $(1)_\lambda$. Moreover, H is a compact homotopy between $H(0,\cdot)\equiv 0$ and $H(1,\cdot)$. In fact, H is compact since j is continuous, $\digamma_\lambda$ is bounded on bounded subsets and L^{-1} is completely continuous. Also, H is usc with closed convex values. Since solutions of $(1)_\lambda$ satisfy $\|x\|_0 \le R^* < R^*+1$ we see that $H(\lambda,\cdot)$ has no fixed points on ∂B_{R^*+1}.

Now, $H(0,\cdot)$ is essential by Theorem 2. Hence H_1 is essential. This implies that $L^{-1}\circ\digamma\circ j$ has a fixed point. Therefore problem (1) has a solution.

This completes the proof of Theorem 4. □

Our next result is based on an application of a fixed point by O'Regan [17].

We shall replace condition (H1) by the following

(H2) $|F(t,x)| \le p(t)\psi(\|x\|)$ for a.e. $t\in I$, all $x\in\mathbb{R}^n$, where $p\in L^1(I,\mathbb{R}_+)$, $\psi:[0,+\infty)\to(0,+\infty)$ is continuous nondecreasing and such that

$$\sup_{\delta\in(0,\infty)} \frac{\delta}{G_0\,\|p\|_{L^1}\,\psi(\delta)} > 1\,.$$

We can state our second result.

Theorem 5. *If the assumption* (H2) *is satisfied, then the boundary value problem* (1) *has at least one solution.*

Proof. This proof is similar to the proof of Theorem 4. Let $M_0>0$ be defined by

$$\frac{M_0}{G_0\,\|p\|_{L^1}\;\psi(M_0)} > 1\,.$$

Let $U := \{x\in AC_0(I); \|x\|_0 < M_0\}$.

Consider the compact operator (see Step 3 above) $\left(L^{-1}\circ\digamma\circ j\right): U\to AC_0(I)$.

Suppose that alternative (A2) in Theorem 3 holds. This means that there exists $x\in\partial U$ such that $x\in\left(L^{-1}\circ\digamma\circ j\right)(x)$, or equivalently

$$x'(t)\in A(t)x(t)+F(t,x(t)) \qquad t\in(0,1),\; Mx(0)+Nx(1)=0\,.$$

Now, as in Step 2 above, assumption (H2) yields

$$\|x(t)\| \le G_0\int_0^1 p(s)\psi(\|x(s)\|)\,ds\,.$$

Since ψ is increasing we get

$$\|x(t)\| \le G_0\int_0^1 p(s)\psi(\|x\|_0)\,ds$$

and, since for $x\in\partial U$ we have $\|x\|_0 = M_0$ this last inequality implies that

$$M_0 \le G_0\int_0^1 p(s)\psi(M_0)\,ds$$

which, in turn gives

$$M_0 \le G_0\Big[\int_0^1 p(s)\,ds\Big]\psi(M_0)\;.$$

Hence,

$$M_0 \le G_0\,\|p\|_{L^1}\,\psi(M_0)$$

This, clearly, contradicts the definition of M_0. Therefore, condition (A2) of Theorem 3 does not hold. Consequently, $L^{-1} \circ F \circ j$ has a fixed point, which is a solution of problem (1). $\square$

We now present a third result based on an inequality of Henry-Bihari type (see [15]). We shall assume that f satisfies

(H3) there exists $p \in L^1(I;\mathbb{R}_+)$ and $\Psi : [0,\infty) \to (0,\infty)$, nondecreasing with the properties

(i) there is $\gamma \in C(I;\mathbb{R}_+)$ such that $e^{-A_0 t}\Psi(u) \leq \gamma(t)\Psi(e^{-A_0 t}u)$ for any $u \geq 0$,

(ii) $\int^{+\infty} \dfrac{d\sigma}{\Psi(\sigma)} = +\infty$, such that $\|F(t,x)\| \leq p(t)\Psi(\|x\|)$ for all $(t,x) \in I \times \mathbb{R}^n$.

As an example of such function Ψ, we can take $\Psi(u) = u^m$, with $0 < m < 1$.

Proposition 1. *Suppose* (H3) *is satisfied. Then there exists* $M_1 > 0$ *such that* $\|x(t)\| \leq M_1$ *for all* $t \in I$ *and any possible solution* x *of* $(1)_\lambda$.

Proof. Let $\langle \cdot,\cdot \rangle$ denote the inner product on $\mathbb{R}^n$. Then, for $f \in S^1_{F(\cdot,x(\cdot))}$ we have $\langle x'(t), x(t)\rangle = \langle A(t)\ x(t) + \lambda f(t), x(t)\rangle$.

Recall that $\langle x'(t), x(t)\rangle = \frac{1}{2}\frac{d}{dt}\|x(t)\|^2$ and use Cauchy-Schwarz inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|^2 \leq \|A(t)\|\,\|x(t)\|^2 + \lambda\,\|f(t)\|\,\|x(t)\|\,.$$

Integrating the above inequality from 0 to 1, we get

$$\|x(t)\|^2 \leq \|x(0)\|^2 + 2A_0\int_0^t \|x(s)\|^2\,ds + 2\int_0^t \|F(s,x(s))\|\,\|x(s)\|\,ds$$

which yields

$$\|x(t)\|^2 \leq \|x(0)\|^2 + 2A_0\int_0^t \|x(s)\|^2\,ds + 2\int_0^t p(s)\Psi(\|x(s)\|)\,\|x(s)\|\,ds\,. \tag{11}$$

Let $u(t) :=$ the righthand side of (11). Then

(i) $\|x(t)\| \leq \sqrt{u(t)}\quad t \in I$;

(ii) $u'(t) = 2A_0\|x(t)\|^2 + 2\ p(t)\Psi(\|x(t)\|)\,\|x(t)\|$.

So that

$$u'(t) \leq 2A_0 u(t) + 2p(t)\Psi\left(\sqrt{u(t)}\right)\sqrt{u(t)}\,. \tag{12}$$

Hence

$$\frac{u'(t)}{2\sqrt{u(t)}} \leq A_0\sqrt{u(t)} + p(t)\Psi\left(\sqrt{u(t)}\right)$$

or

$$\frac{d}{dt}(\sqrt{u(t)}) \leq A_0\sqrt{u(t)} + p(t)\Psi\left(\sqrt{u(t)}\right)\,. \tag{13}$$

Let $v(t) = \sqrt{u(t)}$ for $t \in [0,1]$. Then Inequality (13) becomes

$$v'(t) \leq A_0 v(t) + p(t)\Psi(v(t))$$

or equivalently

$$\left(e^{-A_0 t}v(t)\right)' \leq e^{-A_0 t}p(t)\Psi\left(v(t)\right) . \tag{14}$$

It follows from inequality (14) and the properties of the function Ψ that

$$\left(e^{-A_0 t}v(t)\right)' \leq p(t)\gamma(t)\Psi\left(e^{-A_0 t}v(t)\right) .$$

Let $z(t) = e^{-A_0 t}v(t)$. Then the above inequality gives

$$z'(t) \leq p(t)\gamma(t)\Psi\left(z(t)\right) .$$

Thus

$$\frac{z'(t)}{\Psi\left(z(t)\right)} \leq p(t)\gamma(t) \quad 0 \leq t \leq 1 . \tag{15}$$

Recall that $z(0) = v(0) = \sqrt{u(0)} = \|x\left(0\right)\|$.
Inequality (15) implies that

$$\int_{\|x(0)\|}^{z(t)} \frac{d\sigma}{\Psi\left(\sigma\right)} \leq \int_0^t p(s)\gamma(s)\ ds \leq \|p\|_{L^1}\|\gamma\|_0 . \qquad \square$$

This shows that there exists $M_1 > 0$ such that

$$\|x(t)\| \leq M_1 \quad 0 \leq t \leq 1 .$$

Now, proceeding as in the proof of Theorem 3 we can prove

Theorem 6. *If the assumption* (H3) *is satisfied, then the boundary value problem* (1) *has at least one solution.*

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