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Archivum Mathematicum, Vol. 41 (2005), No. 3, 265--272

Persistent URL: http://dml.cz/dmlcz/107957

ARCHIVUM MATHEMATICUM (BRNO)
Tomus 41 (2005), 265 – 272

COUNTABLE EXTENSIONS OF TORSION ABELIAN GROUPS

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Abstract. Suppose A is an abelian torsion group with a subgroup G such that A/G is countable that is, in other words, A is a torsion countable abelian extension of G. A problem of some group-theoretic interest is that of whether $G \in \mathbb{K}$, a class of abelian groups, does imply that $A \in \mathbb{K}$. The aim of the present paper is to settle the question for certain kinds of groups, thus extending a classical result due to Wallace (J. Algebra, 1981) proved when $\mathbb{K}$ coincides with the class of all totally projective p-groups.

1. Notions, notation and other conventions

In all that follows, let A be an additively written abelian group, G its arbitrary fixed subgroup and $A[p]$ the socle of A consisting of all elements $\{a : pa = 0\}$, p a prime. All other symbols of any character as well as the terminology not explicitly defined herein will follow those from the fundamental monograph books of L. Fuchs [6] and our bibliography [1–5].

It is long known that the second Prüfer theorem, archived in ([6, p. 118, Theorem 18.3]), says that any separable countable abelian p-group is a direct sum of cyclics. We also indicate the well-known generalizations to the last fact that every separable p-primary Σ-group (in particular separable summable or separable totally projective groups), respectively every separable p-primary σ-summable group (in particular separable summable or separable totally projective groups both of limit lengths confinal with ω), is a direct sum of cyclic groups.

A further development on this theme is of Fuchs ([6, p. 24, Proposition 68.3]) who showed that if A is a separable p-group with a basic subgroup B for which A/B is countable, then A is a direct sum of cyclics; however he exploits the Prüfer's affirmation.

As a culmination of a series of such claims, in 1981, Wallace has argued in [12] the following excellent statement-improvement of the preceding one, concerning the extensions of totally projective subgroups by countable factor-groups, namely:

1991 *Mathematics Subject Classification*: 20K10, 20K20, 20K21.

Key words and phrases: countable factor-groups, Σ-groups, σ-summable groups, summable groups, $p^{\omega+n}$-projective groups.

Received September 9, 2003, revised December 1, 2003.

Theorem (Wallace, J. Algebra)**.** *If A is an abelian reduced p-group so that A/G is countable for some totally projective subgroup G of A, then A is totally projective.*

Remark. When $A/G = Z(p^\infty)$, a quasi-cyclic group, this is precisely ([6, volume II, p. 118, Exercise 13]).

For direct sums of cyclic groups we obtain the following consequence.

Corollary. *Suppose A is a separable abelian p-group with a subgroup G such that A/G is countable. Then A is a direct sum of cyclics if and only if so does G.*

Remark. When A/G is separable (equivalently G is nice in A), whence a direct sum of cyclics, the corollary contrasts with an example due to J. Dieudonné on his own criterion (see [6], volume II, p. 16, Exercise 11 or [5] for its generalized version), in which example A/G must be an uncountable and an unbounded direct sum of cyclic groups.

Imitating the Wallace's assertion, it is naturally to pose the following actual in this topic

Problem. Given A is a torsion abelian group and $G \leq A$ such that A/G is countable. If G belongs to any fixed sort of abelian groups, then does this imply that A belongs to the same group sort?

If we presume additional restrictions on G, the query can be simplified. For instance, G must be a direct factor of A in each of the following points:

- G is pure in A and A/G is a direct sum of cyclic groups (Kulikov, [6, p. 143, Theorem 28.2]);
- G is balanced in A and A/G is countable (Megibben, [10, p. 1194, Theorem 2.1] and Fuchs, [6, volume II, p. 99, Exercise 11(a)]);
- G is balanced in A and A/G is a direct sum of countable groups (Hill-Megibben, [9]);
- G is balanced in A and A/G is simply presented (Fuchs, [6, volume II, p. 118, Exercise 11(a)]).

It is of great interest the question which deals with under what circumstances on G, the quotient group A/G being summable or a Σ-group or $p^{\omega+k}$-projective ($k \in \mathbb{N}$) will imply that G is a direct factor of A. The problem is wide-open yet.

It is also a provocation, however, to explore the problem without other serious conditions on G such as being, for example, nice or isotype (respective pure) or balanced, etcetera. At the moment, it is still unknown whether or not they may be dropped. We conjecture that the problem has a negative answer in general, but nevertheless we shall inspect in the sequel its validity (of the type of Prüfer-Fuchs-Wallace) for

(a) torsion Σ-groups;
(b) σ-summable p-groups;
(c) summable p-groups of countable length;
(d) torsion C_λ-groups whenever λ is an arbitrary ordinal;
(e) $p^{\omega+k}$-projective p-groups, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Because for a torsion group A it holds valid that $A = \oplus_p A_p$, where A_p are the p-components for every prime number p, it is enough to bound our attention to p-primary groups.

And so, we come to the central paragraph.

2. Main results

Foremost, for a further freely use, we give some preliminaries on the countable factor-groups.

In fact, assume that A/G is a countable torsion group. Thus we write $A/G = \cup_{n<\omega} (A_n/G)$ where $G \subseteq A_n \subseteq A_{n+1}$ are subgroups of A and, for all naturals n, the factors A_n/G are finite. Consequently, $A_n = \langle a_1^{(n)}, \dots, a_{k_n}^{(n)}, G\rangle$ and $A_n[p] = \langle a_1^{(n)}, \dots, a_{l_n}^{(n)}, G[p]\rangle$ where $0 \leq l_n \leq k_n$ are positive integers or zero. Besides, we observe that $A = \cup_{n<\omega} A_n$ and $A[p] = \cup_{n<\omega} A_n[p]$.

We start with

a) torsion Σ-groups.

The definition and some characteristic properties of such groups can be found in [2] and [3]. However, for the sake of completeness, we shall recall a part of them once again. For this purpose, we denote by H_C a high subgroup of the abelian group C, that is, H_C is maximal with respect to $\cap\, p^\omega C = 0$. So, C is said to be a Σ-group if H_C is a direct sum of cyclic groups.

We begin here with a direct consequence of our results established in [3].

Theorem. *Let C be a balanced subgroup of the abelian p-group A such that A/C is a Σ-group. Then A is a Σ-group if and only if C is a Σ-group.*

Proof. The necessity is well-known even when C is only pure in A.

For the sufficiency, since H_C is pure in H_A and since owing to [3] it holds that $H_{A/C} \cong H_A/H_C$ is a direct sum of cyclics, the above listed classical direct factor statement due to L. Ya. Kulikov assures that $H_A \cong H_C \oplus H_A/H_C$. Thereby, because by assumption H_C is a direct sum of cyclics, it follows at once that so does H_A. Thus, by definition, A is a Σ-group and the proof is completed. $\square$

Remark. It is appeared in ([6, volume II, p. 110, Property A)]) that if G is a balanced subgroup of the abelian p-group A, and both G and A/G are totally projective, then A is totally projective too. Thus our argued above attainment can be considered as a refinement to this fact.

We can decrease the restriction on G being nice and isotype to ordinary purity if we provide A/G is countable. Now, we are ready to formulate

Theorem 1. *Suppose G is a pure subgroup of an abelian p-group A whose quotient A/G is a countable group. Then A is a Σ-group if and only if G is a Σ-group.*

Proof. As we have just noted earlier, the necessity is done.

Next, we treat the more difficult reverse inclusion. Because by hypothesis G is a p-torsion Σ-group, utilizing own group criterion from [2], we write down $G[p] = \cup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ and $G_n \cap p^n G = (p^\omega G)[p]$, for every natural number n.

Furthermore, invoking to the exposition, $A[p] = \cup_{n<\omega}\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}, G_n\rangle$. But $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}\rangle$ are a finite number, hence it is a routine matter to see that there exists a natural number t_n such that $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}\rangle \cap p^{t_n}A \subseteq (p^\omega A)[p]$. Since G is pure in A, we observe that $G \cap p^m A = p^m G$ $\forall m \in \mathbb{N}$ whence $G \cap p^\omega A = p^\omega G$.

Combining these two observations, we compute that $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}, G_n\rangle \cap p^{m_n}A \subseteq (p^\omega A)[p]$ for some $m_n \in \mathbb{N}$, since all problematic elements with (eventually finite) heights $> \max(n, t_n)$ are a finite number of the type $b_n g_n$ for some $b_n \in \langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}\rangle$ and some $g_n \in G_n$ that depends on b_n. Thus, $b_n h_n \in b_n g_n G_n$ for $g_n \neq h_n \in G_n$ and $c_n g_n \in b_n g_n \langle a_1^{(n)}, \cdots, a_{l_n}^{(n)}\rangle$ for $b_n \neq c_n \in \langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}\rangle$ will be with appropriate heights. That is why, the group necessary and sufficient condition of [2] leads us to this that A must be a Σ-group, as promised. The proof is over. □

b) σ-summable p-groups.

Following [1], the reduced abelian p-group A is said to be σ-summable, in the sense of Linton-Megibben, if $A[p] = \cup_{n<\omega}A_n$, $A_n \subseteq A_{n+1}$ and for each $n < \omega$ there is an ordinal $\alpha_n <$ length (A) so that $A_n \cap p^{\alpha_n}A = 0$. From this, it immediately follows that length $(A) \geq \omega$ is confinal with ω.

If K is a subgroup of the σ-summable group A with length $(K) =$ length (A), it follows directly from the definition that K is σ-summable as well.

Referring to the Honda's criterion for summability in countable length (see [6, p. 123, Theorem 84.1] or cf. [10, p. 1194]), it is simple checked that each summable group of countable limit length is σ-summable.

Well, we proceed by proving

Theorem 2. *Suppose A is an abelian reduced p-group of limit length with an isotype subgroup G such that A/G is countable. If G is σ-summable, then so is A.*

Proof. As in the preliminary part of the exposition before point *a*), we derive $A_n[p] = \langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}, G[p]\rangle$ with $L_n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for all $n < \omega$ and $A[p] = \cup_{n<\omega}A_n[p]$.

By assumption, we write down $G[p] = \cup_{n<\omega}G_n$, $G_n \subseteq G_{n+1}$ and $G_n \cap p^{\beta_n}A = 0$ for each $n < \omega$ and some ordinal $\beta_n <$ length (A). Likewise $A[p] = \cup_{n<\omega}\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}, G_n\rangle$ where $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}, G_n\rangle$ is an ascending sequence of subgroups of $A[p]$. For the finite system $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}\rangle$ we obviously deduce that $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}\rangle \cap p^{\gamma_n}A = 0$ for some ordinal $\gamma_n <$ length (A) and every $n < \omega$. Thus $\langle a_1^{(n)}, \ldots, a_{l_n}^{(n)}, G_n\rangle \cap p^{\delta_n}A = 0$ for some $\delta_n <$ length (A). Finally, the definition formulated allows us to conclude that A is indeed σ-summable, as wanted. The proof is finished. □

Remark. When A/G is reduced countable of limit length, hence it is σ-summable, the result also follows from the generalized Dieudonné criterion established by us in [5].

The same attainment is embarrassing for confirmation provided G is not isotype in A.

c) summable p-groups.

For definition of a summable group, the reader can see [8] or [6] where it is given in all details. However, for a convenience of the reader, we shall include it in the text. So, the reduced abelian p-group C of length λ is summable if $C[p] = \oplus_{\alpha<\lambda} C_\alpha$ where, for each ordinal $\alpha < \lambda$, $C_\alpha \setminus \{0\} \subseteq p^\alpha C \setminus p^{\alpha+1} C$.

Theorem 3. *Suppose G is an isotype subgroup of the reduced abelian p-group A such that A/G is countable. Then*

(*) *A is summable of countable length if and only if G is summable of countable length.*

(**) *A is summable if and only if G is summable, provided G is nice in A.*

Proof. (*) First of all, choose G to be summable. Owing to the Honda's criterion on summability of countable length (see, for instance, [6, p. 123, Theorem 84.1]; [8] or [10]), $G[p] = \cup_{n<\omega} G_n$, $G_n \subseteq G_{n+1}$ and all G_n are height-finite in G whence in A because of the isotypity of G in A. Taking into account the previous discussion from the exposition, $A_n[p] = \langle a_1^{(n)}, \dots, a_{l_n}^{(n)}, G[p] \rangle = \cup_{i<\omega} \langle a_1^{(n)}, \dots, a_{l_n}^{(n)}, G_i \rangle$ where the system $\{a_1^{(n)}, \dots, a_{l_n}^{(n)}, G_i | L_n \geq 0\}$ forms an ascending chain of subgroups for each index $n < \omega$ and each index $i < \omega$, and $A[p] = \cup_{n<\omega} A_n[p]$. We therefore obtain that $A[p] = \cup_{n<\omega} \langle a_1^{(n)}, \dots, a_{l_n}^{(n)}, G_n \rangle$, where it is apparent to verify that all $\langle a_1^{(n)}, \dots, a_{l_n}^{(n)}, G_n \rangle$ are with finite height-spectrum in A. Henceforth, we really obscure by the cited above Honda's criterion that A should be summable.

Let now A be summable. We wish to apply ([8] or [6, p. 125, Proposition 84.4]) to get that G is summable.

(**) If now G is nice in A, whence it is balanced in A, conforming with the above stated Megibben's claim on the direct factor, we infer that G is a direct factor of A, so everything is fulfilled via [6] and [8].

The theorem is proved in general after all. □

Analysis. The examination involves when G is of the uncountable length $\omega_1 = \Omega$ or when G is not a direct factor of A. In that direction, whether or not A being summable plus A/G being countable with G isotype in A do imply that G is summable?

According to the preceding discussion in the exposition, $A = \cup_{n<\omega} A_n$ and $A_n \subseteq A_{n+1}$. Moreover, all A_n/G are finite and if we assume G is pure in A, hence in A_n, consulting with the above listed classical direct factor statement of Kulikov, we establish $A_n = G \oplus F_n$ for some finite $F_n \leq A_n$ and for every $n < \omega$. So, all A_n are summable, but perhaps A need not be summable provided length $(A) = \omega_1$ or A_n are not isotype in A (see [7]).

We also emphasize that $A[p] \cong G[p] \oplus (A/G)[p]$ and $(p^\alpha A)[p] \cong (p^\alpha G)[p] \oplus \big((p^\alpha A + G)/G\big)[p]$ for every $\alpha \geq 1$.

We continue with an observation on Theorem 3.

Example. The condition on A being reduced, however, cannot be removed. In fact, select an abelian p-group A such that $A/p^\omega A$ is a direct sum of cyclics, $p^\omega A$ is divisible (or more generally, $p^\omega A$ is an unreduced group) and $(p^\omega A)[p]$ is infinite countable. Therefore A is a Σ-group but not a summable group. Of course, $p^\omega A$ is countable divisible with countable rank and thus by virtue of a result due to B. Charles - Ch. Megibben, A is a direct sum of a countable group and of a direct sum of cyclic groups. On the other hand, H_A is a direct sum of cyclics. Because the subgroup H_A is pure in A, the direct decomposition $A[p] = H_A[p] \oplus (p^\omega A)[p]$ guarantees that $(A/H_A)[p] \cong A[p]/H_A[p] \cong (p^\omega A)[p]$ is infinite countable whence so is the divisible A/H_A. Now, bearing in mind that H_A is isotype in A (cf. [6]), we are done.

Remark. We mention that A/G is not necessarily reduced in the first part (*) of our theorem, whereas in the second one (**) it is indebted to be reduced since G is nice in A. Indeed, for instance, we can put $G = H_A$. First, let $p^\omega A$ be infinite. Since $p^\omega A$ is reduced as a subgroup of the reduced A, we infer that $|p^\omega A| = |(p^\omega A)[p]| \geq \aleph_0$ and consequently A/H_A countable yields as above that $p^\omega A$ is at most countable hence summable. That is why, referring to [4], H_A summable (i.e. a direct sum of cyclics) implies A is summable of countable length. If, in the remaining case, $p^\omega A$ is finite whence summable, again [4] ensures our claim.

d) torsion C_λ-groups, λ an ordinal.

The definition of a C_λ-group, first introduced by Megibben, states as follows: The abelian p-group M is called a C_λ-group if $M/p^\alpha M$ is totally projective for any ordinal $\alpha < \lambda$. It is well-known that every totally projective p-group is a C_λ-group ([6]). The major result here asserts thus.

Theorem 4. *Let A be an abelian p-group with an isotype subgroup G which is a C_λ-group for an arbitrary ordinal number λ and let A/G be countable. Then A is a C_λ-group.*

Proof. In virtue of the assumptions, $G/p^\alpha G$ is totally projective for each $\alpha < \lambda$. Since G is isotype in A, we detect that $G/p^\alpha G = G/(G \cap p^\alpha A) \cong (G + p^\alpha A)/p^\alpha A$ may be interpreted as a subgroup of $A/p^\alpha A$. Moreover, $A/p^\alpha A/(p^\alpha A + G)/p^\alpha A \cong A/(p^\alpha A + G)$ is also countable as an epimorphic image of the countable A/G. Thus, we see that all conditions from the Wallace's theorem are satisfied, hence it may be exploited to get the desired claim that $A/p^\alpha A$ is totally projective for every $\alpha < \lambda$. So, the proof is complete. $\square$

Remark. It is known that every isotype subgroup of a C_Ω-group is also a C_Ω-group (mainly by P. Hill – see, for example, cf. [10]), but in the general situation even a balanced subgroup of a totally projective p-group need not be totally projective.

e) $p^{\omega+k}$-projective p-groups, $k \in \mathbb{N} \cup \{0\}$.

This section is devoted to the exploration of so-called p^δ-projective groups, δ an ordinal ($\delta = \omega + k$ in our case), introduced by Nunke in [13]. We require only for information a few more comprehensive characterizations (see [6]): An abelian p-group A is totally projective $\Leftrightarrow$ $A/p^\delta A$ is p^δ-projective $\forall\delta$; the totally projective p-group A is p^δ-projective $\Leftrightarrow$ length $(A) \leq \delta$; if A is p^δ-projective $\Rightarrow$ length $(A) \leq \delta$.

In order to prove the corresponding theorem for $p^{\omega+k}$-projective groups, we recognize for the convenience of the reader and for the sake of completeness the following criterion given in the more potent original formulation. To simplify below the term, we denote $p^\omega A = A^1$.

Criterion (Nunke, [13])**.** Let E be an abelian p-group. Then E is $p^{\omega+k}$-projective for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ if and only if there is $T \leq E[p^k]$ with the property E/T is a direct sum of cyclic groups.

We also need the following simple but however useful

Lemma. *Let M/T be a separable p-group and T a pure separable p-subgroup of the abelian group M. Then M is separable.*

Proof. Since $0 = (M/T)^1 = (M^1 + T)/T$, it occurs that $T \supseteq M^1$, hence $M^1 = T^1 = 0$, as claimed. This ends the proof. □

We are now in a position to proceed by proving

Theorem 5. *Suppose A is an abelian reduced p-group of length not exceeding $\omega+k$ for some nonnegative integer k with a pure and nice subgroup G such that A/G is countable. Then A is $p^{\omega+k}$-projective if and only if G is $p^{\omega+k}$-projective.*

Proof. It easily follows from the Nunke's criterion the plain fact that a subgroup of a $p^{\omega+k}$-projective group inherits the same property.

Next, we deal with the more hard converse implication. Consulting with the above formulated Nunke's necessary and sufficient condition, there exists $C \subseteq G[p^k]$ so that G/C is a direct sum of cyclics, hence $G^1 \subseteq C$. To confirm that A is $p^{\omega+k}$-projective too, it suffices again by virtue of the Nunke's criterion to find a subgroup K of $A[p^k]$ with the property A/K is a direct sum of cyclic groups. This is accomplished by showing that $A/(C+A^1)$ is a direct sum of cyclics where we set $K = C + A^1 \subseteq A[p^k]$ since we obviously have $A^1 \subseteq A[p^k]$. In fact, $(G + A^1)/(C + A^1) \cong G/\big(G \cap (C + A^1)\big) = G/\big(C + (G \cap A^1)\big) = G/(C + G^1) = G/C$ is a direct sum of cyclic groups. On the other hand we compute, $[(G + A^1)/(C + A^1)] \cap p^n\big(A/\big(C + A^1\big)\big) = [(G + A^1)/(C + A^1)] \cap \big((C + p^n A)/(C + A^1)\big) = [(G + A^1) \cap (C + p^n A)]/(C + A^1) = \big(C + [(G + A^1) \cap p^n A]\big)/(C + A^1) = [(C + A^1) + (G \cap p^n A)]/(C + A^1) = (C + A^1 + p^n G)/(C + A^1) = p^n[(G + A^1)/(C + A^1)]$, hence by definition $A/(C + A^1)$ contains $(G + A^1)/(C + A^1)$ as a pure subgroup. Moreover, G being nice in A yields that so is $G + A^1$ in A and therefore $A/(C + A^1)/(G + A^1)/(C + A^1) \cong A/(G + A^1)$ is both separable and countable as an epimorphic image of the countable A/G. Thus, by what we have just shown above, in view of the Lemma and of the Wallace's criterion, $A/(C + A^1)$ is a direct sum of cyclics, as stated. The proof of the theorem is completed. □

Remark. Since G may be pure but no isotype in A, it is not necessarily its direct factor. Moreover, since A/G possesses length less than or equal to $\omega + k$, it follows with the aid of [6] that this quotient is $p^{\omega+k}$-projective.

We close the study with

3. Concluding commentary

The author of this article feels that the extra restrictions on G being pure, isotype, balanced etc. can be ignored in the current situation, although concrete examples to this conjecture are not extracted yet.

About the mixed case (i.e. considering mixed abelian groups), the problem seems to be very difficult. In that aspect, Ch. Megibben probably had established in [11] that the Wallace's theorem may be strengthened to the class of mixed simply presented groups, as the title of the paper [11] shows. We have not seen the Megibben's work, so in this way does his idea remain valid even for global Warfield groups?

As a final discussion, we think that the obtained above assertions hold true and for the kinds of *N-groups*, *S-groups*, *A-groups* and *IT-groups*, that are larger classes of torsion groups than the totally projective ones.

Acknowledgements. The author owes deep thanks to the anonymous specialist referee for the warm comments and helpful suggestions made, and to the Editor Prof. Jan Trlifaj for the expert editorial advice.

References

[1] Danchev, P. V., *Commutative group algebras of σ-summable abelian groups*, Proc. Amer. Math. Soc. (9) **125** (1997), 2559–2564.

[2] ———, *Commutative group algebras of abelian Σ-groups*, Math. J. Okayama Univ. **40** (1998), 77–90.

[3] ———, *Commutative group algebras of highly torsion-complete abelian p-groups*, Comment. Math. Univ. Carolin. (4) **44** (2003), 587–592.

[4] ———, *Commutative group algebras of summable abelian p-groups*, Comm. Algebra, in press.

[5] ———, *Generalized Dieudonné criterion*, Acta Math. Univ. Comenian. (1) **74** (2005), 15–24.

[6] Fuchs, L., *Infinite Abelian Groups*, Volumes **I** and **II**, Mir, Moskva, 1974 and 1977. (In Russian.)

[7] Hill, P. D., *Criteria for freeness in groups and valuated vector spaces*, Lect. Notes in Math. **616** (1977), 140–157.

[8] Hill, P. D. and Megibben, C. K., *On direct sums of countable groups and generalizations*, Études sur les Groupes Abéliens, Paris (1968), 183–206.

[9] ———, *Extending automorphisms and lifting decompositions in abelian groups*, Math. Annalen **175** (1968), 159–168.

[10] Megibben, C. K., *The generalized Kulikov criterion*, Canad. J. Math. **21** (1969), 1192–1205.

[11] ———, *Countable extensions of simply presented groups*, Internet information.

[12] Wallace, K. D., *On mixed groups of torsion-free rank one with totally projective primary components*, J. Algebra **17** (1971), 482–488.

[13] Nunke, R. J., *Purity and subfunctors of the identity*, Topics in Abelian Groups, Scott Foresman and Co., (1963), 121–171.

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