Zhongkui Liu
On $S$-Noetherian rings

Archivum Mathematicum, Vol. 43 (2007), No. 1, 55--60

Persistent URL: http://dml.cz/dmlcz/108049

Terms of use:

© Masaryk University, 2007

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ON S-NOETHERIAN RINGS

Liu Zhongkui

Abstract. Let $R$ be a commutative ring and $S \subseteq R$ a given multiplicative set. Let $(M, \leq)$ be a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$. Then it is shown, under some additional conditions, that the generalized power series ring $[[R^M, \leq]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and $M$ is finitely generated.

1. Introduction

Let $R$ be a commutative ring and $S \subseteq R$ a given multiplicative set. According to [2], an ideal $I$ of $R$ is called $S$-finite if $sI \subseteq J \subseteq I$ for some $s \in S$ and some finitely generated ideal $J$. $R$ is called $S$-Noetherian if each ideal of $R$ is $S$-finite. Clearly every Noetherian ring is $S$-Noetherian for any multiplicative set $S$.

Let $X_1, \ldots, X_n$ be indeterminates. It was showed in [2], Proposition 10, that if $S \subseteq R$ is an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors and $R$ is $S$-Noetherian, then $R[[X_1, \ldots, X_n]]$ is $S$-Noetherian. It was proved in [3], Theorem 4.3, that if $(M, \leq)$ is a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$, then the generalized power series ring $[[R^M, \leq]]$ is left Noetherian if and only if $R$ is left Noetherian and $M$ is finitely generated. By the technique developed in [3] we show that if $R$ is $S$-Noetherian and $M$ is finitely generated.

Throughout this note all rings are commutative with identity and all monoids are commutative. Any concept and notation not defined here can be found in [2], [3] and [6].

2. Generalized power series rings

Let $(M, \leq)$ be an ordered set. Recall that $(M, \leq)$ is artinian if every strictly decreasing sequence of elements of $M$ is finite, and that $(M, \leq)$ is narrow if every...
subset of pairwise order-incomparable elements of $M$ is finite. Let $M$ be a commutative monoid. Unless stated otherwise, the operation of $M$ shall be denoted additively, and the neutral element by 0.

Let $(M, \leq)$ be a strictly ordered monoid (that is, $(M, \leq)$ is an ordered monoid satisfying the condition that, if $m_1, m_2, m \in M$ and $m_1 < m_2$, then $m_1 + m < m_2 + m$), and $R$ a ring. Let $[R^{M, \leq}]$ be the set of all maps $f : M \rightarrow R$ such that $\operatorname{supp}(f) = \{m \in M \mid f(m) \neq 0\}$ is artinian and narrow. With pointwise addition, $[R^{M, \leq}]$ is an abelian additive group. For every $m \in M$ and $f, g \in [R^{M, \leq}]$, let $X_m(f, g) = \{(u, v) \in M \times M \mid m = u + v, f(u) \neq 0, g(v) \neq 0\}$. It follows from [9], 1.16, that $X_m(f, g)$ is finite. This fact allows us to define the operation of convolution:

$$(fg)(m) = \sum_{(u, v) \in X_m(f, g)} f(u) g(v).$$

With this operation, and pointwise addition, $[R^{M, \leq}]$ becomes a commutative ring, which is called the ring of generalized power series. The elements of $[R^{M, \leq}]$ are called generalized power series with coefficients in $R$ and exponents in $M$.

For example, if $M = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[R^{\mathbb{N} \cup \{0\}, \leq}] \cong R[[x]]$, the usual ring of power series. If $M$ is a commutative monoid and $\leq$ is the trivial order, then $[R^{M, \leq}] = R[M]$, the monoid-ring of $M$ over $R$. Further examples are given in [5] and [6]. Results for rings of generalized power series appeared in [3], [5]-[11].

Any monoid $M$ has the algebraic or natural preorder defined by $a \preceq b$ if $a + c = b$ for some $c \in M$. In general, $a \preceq b \preceq a$ does not imply $a = b$, so $\preceq$ is not always a partial order on $M$. The symbol $\preceq$ will always be used for the algebraic preorder of a monoid in this paper.

Recall from [3] that if $(M, \leq)$ and $(N, \leq)$ are ordered monoids, then a strict monoid homomorphism $\sigma : (M, \leq) \rightarrow (N, \leq)$ is a monoid homomorphism $\sigma : M \rightarrow N$ which is strictly increasing with respect to the partial orders $\leq$.

**Lemma 2.1.** Let $(M, \leq)$, where $|M| > 1$, be a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$. Then for some commutative free monoid $F$, there exists a surjective strict monoid homomorphism $\sigma : (F, \preceq) \rightarrow (M, \preceq)$.

**Proof.** It follows from [3], Lemma 3.1 and Lemma 3.2. \qed

Note from the proof of [3], Lemma 3.2, that if $M$ is finitely generated, then the free monoid $F$ can be chosen finitely generated.

**Lemma 2.2.** Let $\alpha : R \rightarrow R'$ be a surjective ring homomorphism and $S \subseteq R$ a multiplicative set of $R$. If $R$ is $S$-Noetherian, then $R'$ is $\alpha(S)$-Noetherian.

**Proof.** It follows from the definition. \qed

Let $m \in M$. We define a mapping $e_m \in [R^{M, \leq}]$ as follows:

$$e_m(m) = 1, \quad e_m(x) = 0, \quad m \neq x \in M.$$
Let $r \in R$. Define a mapping $c_r \in \mathbb{[R^{M, \leq}]}$ as follows:

$$c_r(0) = r, \quad c_r(m) = 0, 0 \neq m \in M.$$ 

Then $R$ is isomorphic to the subring $\{c_r \mid r \in R\}$ of $\mathbb{[R^{S, \leq}]}$. Thus if $S$ is a multiplicative set of $R$ then $C(S) = \{c_r \mid r \in S\}$ is a multiplicative set of $\mathbb{[R^{M, \leq}]}$. In the following we will say $\mathbb{[R^{M, \leq}]}$ is S-Noetherian if $\mathbb{[R^{M, \leq}]}$ is $C(S)$-Noetherian.

It was proved in [3], Theorem 4.3, that if $(M, \leq)$ satisfies the condition that $0 \leq m$ for every $m \in M$, then $\mathbb{[R^{M, \leq}]}$ is left Noetherian if and only if $R$ is left Noetherian and $M$ is finitely generated. For $S$-Noetherian rings we have the following result. Recall from [1] that a multiplicative set $S$ of a ring $R$ is said to be anti-Archimedean if $(\cap_{n \geq 1} s^n R) \cap S \neq \emptyset$ for every $s \in S$. Clearly every multiplicative set consisting of units is anti-archimedean.

**Theorem 2.3.** Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Let $(M, \leq)$ be a strictly ordered monoid satisfying the condition that $0 \leq m$ for every $m \in M$. Then $\mathbb{[R^{M, \leq}]}$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and $M$ is finitely generated.

**Proof.** We complete the proof by adapting the proof of [3], Theorem 4.3. Suppose that $\mathbb{[R^{M, \leq}]}$ is $S$-Noetherian. Let $\{m_n \mid n \in \mathbb{N}\}$ be an infinite sequence in $M$. We will show that there exist $i < j$ in $\mathbb{N}$ such that $m_i \leq m_j$. Consider the ascending chain of ideals of $\mathbb{[R^{M, \leq}]}$: $\mathbb{[R^{M, \leq}]}e_{m_1} \subseteq \mathbb{[R^{M, \leq}]}e_{m_1} + \mathbb{[R^{M, \leq}]}e_{m_2} \subseteq \cdots \subseteq \mathbb{[R^{M, \leq}]}e_{m_1} + \cdots + \mathbb{[R^{M, \leq}]}e_{m_i} \subseteq \cdots$. Denote that $I = \sum_{i=1}^{\infty} (\mathbb{[R^{M, \leq}]}e_{m_1} + \cdots + \mathbb{[R^{M, \leq}]}e_{m_i})$. Then $I$ is an ideal of $\mathbb{[R^{M, \leq}]}$. Since $\mathbb{[R^{M, \leq}]}$ is $S$-Noetherian, there exist $s \in S$ and a finitely generated ideal $J$ of $\mathbb{[R^{M, \leq}]}$ such that $c_s I \subseteq J \subseteq I$. Clearly there exists an integer $k$ such that $J \subseteq \mathbb{[R^{M, \leq}]}e_{m_1} + \cdots + \mathbb{[R^{M, \leq}]}e_{m_k}$. Thus $c_s e_{m_{k+1}} = f_1 e_{m_1} + f_2 e_{m_2} + \cdots + f_k e_{m_k}$ for some $f_1, f_2, \ldots, f_k \in \mathbb{[R^{M, \leq}]}$. Hence $m_{k+1} \in \bigcup_{i=1}^{k} \text{supp}(f_i e_{m_i}) \subseteq \bigcup_{i=1}^{k} \text{supp}(f_i) + m_i$. This implies that $m_{k+1} = t + m_i$ for some $i < k+1$ and $t \in M$. Thus $m_i \leq m_{k+1}$. Hence we have shown that for any infinite sequence $\{m_n \mid n \in \mathbb{N}\}$ in $M$ there exist $i < j$ in $\mathbb{N}$ such that $m_i \leq m_j$. Thus, by ([3], Lemma 3.3), $M$ is finitely generated.

Let

$$W = \{f \in \mathbb{[R^{M, \leq}]} \mid f(0) = 0\}.$$ 

For any $f \in W$ and any $g \in \mathbb{[R^{M, \leq}]}$,

$$(gf)(0) = \sum_{(u,v) \in \mathbb{[R^{M, \leq}]}e_{n(u,g,f)}} g(u) f(v) = g(0) f(0) = 0,$$

which implies that $gf \in W$. Similarly $fg \in W$. Now it is easy to see that $W$ is an ideal of $\mathbb{[R^{M, \leq}]}$. Define a mapping $\alpha : R \rightarrow \mathbb{[R^{M, \leq}]} / W$ via

$$\alpha(r) = c_r + W, \quad \forall r \in R.$$ 

Clearly $\alpha$ is a homomorphism of rings. For any $f \in \mathbb{[R^{M, \leq}]}$, $f + W = c_f(0) + W = \alpha(f(0))$, which implies that $\alpha$ is an epimorphism. Clearly $\alpha$ is a monomorphism. Thus there is an isomorphism of rings $R \cong \mathbb{[R^{M, \leq}]} / W$. Now it follows from Lemma 2.2 that $R$ is $S$-Noetherian.
Remark 2.4. Note that the direct implication in Theorem 2.3 holds without further assumptions on $S$. But the following example (see [2]) shows that the assumptions on $S$ is needed for the converse. Let $(V, M)$ be a rank-one nondiscrete valuation domain. Then $V$ is $S$-Noetherian where $S = V - \{0\}$, but $V[[x]]$ is not $S$-Noetherian by [2]. In fact, $V[[x]]_S$ is not Noetherian by part (3) of [4], Theorem 3.13.

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a numerical monoid. It is well-known that any numerical monoid is finitely generated (see 1.3 of [6]). Thus we have the following result.

**Corollary 2.5.** Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Let $M$ be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N} \cup \{0\}$. Then $[[R^M, \leq]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian.

Let $p_1, \ldots, p_n$ be prime numbers. Set

$$N(p_1, \ldots, p_n) = \{p_1^{m_1}p_2^{m_2}\cdots p_n^{m_n} | m_1, m_2, \ldots, m_n \in \mathbb{N} \cup \{0\}\}.$$ 

Then $N(p_1, \ldots, p_n)$ is a submonoid of $(\mathbb{N}, \cdot)$. Let $\leq$ be the usual natural order.

**Corollary 2.6.** Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Then the ring $[[R^{N(p_1,\ldots,p_n), \leq}]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian.

**Corollary 2.7.** Let $(M_1, \leq_1), \ldots, (M_n, \leq_n)$ be strictly ordered monoids satisfying the condition that $0 \leq_i m_i$ for every $m_i \in M_i$. Denote by $(\text{lex } \leq)$ the lexicographic order on the monoid $M_1 \times \cdots \times M_n$. Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors. Then the following statements are equivalent.

1. The ring $[[R^{M_1 \times \cdots \times M_n}(\text{lex } \leq)]]$ is $S$-Noetherian.
2. $R$ is $S$-Noetherian and each $M_i$ is finitely generated.

**Proof.** It is easy to see that $(S_1 \times \cdots \times S_n, (\text{lex } \leq))$ is a strictly ordered monoid and $(0, \ldots, 0)(\text{lex } \leq)(m_1, \ldots, m_n)$ for each $(m_1, \ldots, m_n) \in M_1 \times \cdots \times M_n$. Thus, by Theorem 2.3, $[[R^{M_1 \times \cdots \times M_n}(\text{lex } \leq)]]$ is $S$-Noetherian if and only if $R$ is $S$-Noetherian and each $M_i$ is finitely generated.
Let $X_1, \ldots, X_n$ be indeterminates. It was showed in [2], Proposition 10 that if $S \subseteq R$ is an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors and $R$ is $S$-Noetherian, then $R[[X_1, \ldots, X_n]]$ is $S$-Noetherian. For Laurent series rings we have a same result.

**Theorem 3.1.** Let $R$ be a ring and $S \subseteq R$ an anti-Archimedean multiplicative set of $R$ consisting of nonzerodivisors and $X$ an indeterminate. If $R$ is $S$-Noetherian, then so is $R[[X, X^{-1}]]$.

**Proof.** Let $A$ be an ideal of $R[[X, X^{-1}]]$. We will show that $A$ is $S$-finite. For any $0 \neq f \in R[[X, X^{-1}]]$, we denote by $\pi(f)$ the smallest integer $k$ such that $f(k) \neq 0$. For every $k \in \mathbb{Z}$, set

$$I_k = \{ f(k) \mid f \in A, \pi(f) = k \},$$

and $I = \bigcup_{k \in \mathbb{Z}} I_k$. Let $J$ be the ideal of $R$ generated by $I$. Since $R$ is $S$-Noetherian, there exist $w \in S$, $f_1, \ldots, f_m \in A$ such that $wJ \subseteq \sum_{i=1}^{m} f_i(k_i)R$, where $k_i = \pi(f_i)$, $i = 1, \ldots, m$.

Consider any $0 \neq f \in A$. Suppose that $\pi(f) = k$. Then there exist $r_{ik} \in R$ such that $wf(k) = \sum_{i=1}^{m} f_i(k_i)r_{ik}$. Set $g_{k+1} = wf - \sum_{i=1}^{m} f_i X^{k-k_i}r_{ik}$. Then $\pi(g_{k+1}) \geq k+1$. Clearly $g_{k+1} \in A$. Thus there exist $r_{i,k+1} \in R$, $i = 1, \ldots, m$, such that $wg_{k+1}(k+1) = \sum_{i=1}^{m} f_i(k_i)r_{i,k+1}$. Set $g_{k+2} = wg_{k+1} - \sum_{i=1}^{m} f_i X^{k+1-k_i}r_{i,k+1}$. Then $\pi(g_{k+2}) \geq k+2$. Continuing in this manner, for any $n > 0$, we get $r_{i,k+n} \in R$ and $g_{k+n} \in A$ such that $g_{k+n+1} = wg_{k+n} - \sum_{i=1}^{m} f_i X^{k+n-k_i}r_{i,k+n}$ and $\pi(g_{k+n}) \geq k+n$. Thus

$$w^n f = w^{n-1} g_{k+1} + w^{n-1} \sum_{i=1}^{m} f_i X^{k-k_i} r_{i,k}$$

$$= \cdots = g_{k+n} + \sum_{j=1}^{n} \sum_{i=1}^{m} f_i X^{k+j-1-k_i} w^{n-j} r_{i,k+j-1}$$

$$= g_{k+n} + \sum_{i=1}^{m} f_i \left( \sum_{j=1}^{n} X^{k+j-1-k_i} w^{n-j} r_{i,k+j-1} \right).$$

Since $S$ is anti-Archimedean, there exists $t \in (\cap w^j R) \cap S$. Thus $t = w^j r_j$ for some $r_j \in R$. Since $w$ is a nonzerodivisor, we have $r_n w^{n-j} = r_j$ for $j \leq n$. So

$$tf = r_n g_{k+n} + \sum_{i=1}^{m} f_i \left( \sum_{j=1}^{n} X^{k+j-1-k_i} r_j r_{i,k+j-1} \right).$$

Now it is easy to see that

$$tf = \sum_{i=1}^{m} f_i \left( \sum_{j=1}^{\infty} X^{k+j-1-k_i} r_j r_{i,k+j-1} \right) \in \sum_{i=1}^{m} f_i R[[X, X^{-1}]].$$

Hence $tA \subseteq \sum_{i=1}^{m} f_i R[[X, X^{-1}]]$. Consequently, $R[[X, X^{-1}]]$ is $S$-Noetherian.

**Acknowledgement.** The author would like to express his sincere thanks to the referee for valuable suggestions. This work was supported by National Natural
Science Foundation of China, TRAPOYT and the Cultivation Fund of the Key Scientific and Technical Innovation Project, Ministry of Education of China.

REFERENCES


Department of Mathematics, Northwest Normal University
Lanzhou 730070, Gansu, People’s Republic of China
E-mail: liuzk@nwnu.edu.cn