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SPECTRAL PROPERTIES OF A CERTAIN CLASS OF CARLEMAN OPERATORS

S. M. Bahri

Abstract. The object of the present work is to construct all the generalized spectral functions of a certain class of Carleman operators in the Hilbert space $L^2(X, \mu)$ and establish the corresponding expansion theorems, when the deficiency indices are (1,1). This is done by constructing the generalized resolvents of $A$ and then using the Stieltjes inversion formula.

1. Preliminaries

The set of generalized resolvents of a symmetric operator $A$ with defect indices $(1, 1)$ was first derived independently by Naimark [15] and Krein [10]. The case of defect indices $(m, m)$, $m \in \mathbb{N}$ is due to Krein [11]. Saakjan [19] extended Krein’s formula to the general case of defect indices $(m, m)$, $m \in \mathbb{N} \cup \{\infty\}$. In another form, the generalized resolvent formula for symmetric operators (including the case of non-densely defined operators) has been obtained by Straus [20, 21].

Let $H$ be a Hilbert space endowed with the inner product $(\cdot, \cdot)$, and let $A$: $D(A) \subset H \longrightarrow H$ be a densely defined closed linear operator whose range is denoted $R(A)$.

1.1. Basic Spectral Properties. We say that $\lambda \in \mathbb{C}$ is a regular point of the operator $A$ if the resolvent $R_\lambda = (A - \lambda I)^{-1}$ exists and is a bounded operator defined everywhere in $H$ ($I$ denotes the identity operator in $H$). In this case we say that $\lambda$ belongs to $\rho(A)$, the resolvent set of $A$. $R_\lambda$ is an analytic operator function of $\lambda$ on $\rho(A)$. The number $\lambda \in \mathbb{C}$ is said to be an eigenvalue of $A$ if there exists an $f \in D(A)$ for which $f \neq 0$ and $Af = \lambda f$. In this case, the operator $A - \lambda I$ is not injective, i.e., $\ker(A - \lambda I) \neq \{0\}$. The complement of $\rho(A)$, in the complex plane, is denoted by $\sigma(A)$ and is called the spectrum of $A$.

A resolution of the identity [1] is a one-parameter family $\{E_t\}$, $-\infty < t < \infty$, of orthogonal projection operators acting on a Hilbert space $H$, such that

1) $E_s \leq E_t$ if $s \leq t$ (monotonicity);

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ii) $E_t$ is strongly left continuous, i.e. $E_{t-0} = E_t$ for every $t \in \mathbb{R}$;

iii) $E_t \xrightarrow{\text{a.s.}} 0$ as $t \to -\infty$ and $E_t \xrightarrow{\text{a.s.}} I$ as $t \to \infty$; here 0 and $I$ are the zero and the identity operator on the space $H$.

Condition ii) can be replaced by the condition of strong right continuity at every point $t \in \mathbb{R}$.

From this it follows that, for each fixed $f \in H$, the function $\rho_f : \mathbb{R} \to [0,1)$ given by

$$
(1.1) \quad \rho_f (t) = (E(t)f, f) = \|E(t)f\|^2
$$

is is bounded, non-decreasing, left continuous and

$$
(1.2) \quad \lim_{t \to -\infty} \rho_f (t) = \|f\|^2, \quad \lim_{t \to -\infty} \rho_f (t) = 0.
$$

In [1] is proven that for each resolution of the identity $E_t$ ($-\infty \leq t \leq +\infty$) corresponds a uniquely defined self adjoint operator $\hat{A}$, admitting the following integral representation

$$
(1.3) \quad \hat{A} = \int_{-\infty}^{+\infty} t dE_t,
$$

where the integral is understood as the strong limit of the integral sums for each $f \in D(\hat{A})$, and

$$
(1.4) \quad D(\hat{A}) = \left\{ f : \int_{-\infty}^{+\infty} t^2 d(E_t f, f) < \infty \right\}
$$

is satisfied. The resolvent $\hat{R}_\lambda$ and the spectral function $E_t$ of a self adjoint operator $\hat{A}$ are bound by the relation

$$
(1.5) \quad \hat{R}_\lambda = \int_{-\infty}^{+\infty} \frac{dE_t}{t-\lambda}, \quad \lambda \in \rho(\hat{A}),
$$

in the sense of strong limit.

The resolution of the identity given by the operator $\hat{A}$ completely determines the spectral properties of that operator, namely:

$\alpha$) a real number $t_0$ is a regular point of $A$ if and only if it is a point of constancy, that is, if there is an $\varepsilon > 0$ such that $E_{t_0+\varepsilon} - E_{t_0-\varepsilon} = 0$;

$\beta$) a real number $t_0$ is an eigenvalue of $A$ if and only if $\lambda$ is a jump point of $E_t$, that is, $E_{t_0+0} - E_{t_0} \neq 0$.

Hence the resolution of the identity determined by the operator is also called the spectral function of this operator.

1.2. **Deficiency indices.** The defect number is the dimension of the orthogonal complement to $R(A)$

$$
d_A = \dim (H \ominus R(A)) = \dim \ker (A^*),
$$

where $A^*$ is the adjoint operator of $A$ and $\ker (A^*) = \{ f \in D(A^*) : A^* f = 0 \}$, $D(A^*)$ being the domain of $A^*$.
Let \( A \) be a symmetric operator, \( \tilde{A} \) its symmetric extension, then the following relation holds
\[
(1.6) \quad A \subset \tilde{A} \subset \tilde{A}^* \subset A^*.
\]
The interest of (1.6) resides in the following conclusion: all symmetrical extension of \( A \) comes of a restriction of the domain of \( A^* \). So \( D(\tilde{A}) \) is a subspace between \( D(A) \) and \( D(A^*) \). To construct the extensions \( \tilde{A} \) it is therefore well to examine the structure of the space \( D(A^*) \). Let’s put
\[
N_{\lambda} = \ker (A^* - \lambda I) \quad \text{and} \quad \bar{N}_{\lambda} = \ker (A^* - \bar{\lambda} I), \quad (\Im m \lambda > 0),
\]
with respective dimensions \( n_+ \), \( n_- \). They are called the deficiency indices of the operator \( A \) and will be denoted by the ordered pair \((n_+, n_-)\). It being, in the Hilbert space \( D(A^*) \) we have the following hilbertienne decomposition [4]
\[
(1.7) \quad D(A^*) = D(A) \oplus N_{\lambda} \oplus \bar{N}_{\lambda}.
\]
A possesses self adjoint extensions [6] if and only if \( n_+ = n_- \). We get in this case all self adjoints extensions of \( A \) from all isometric Cayley transforms \( V = (A - \lambda I)(A - \bar{\lambda} I)^{-1} \) defined from \( \bar{N}_{\lambda} \) to \( N_{\lambda} \).

1.3. Generalized resolvents formulas. In the general case, every symmetric operator \( A \) can be prolonged in a selfadjoint operator \( A^+ \) defined in a wide space \( H^+ \) containing \( H \). If \( E_t^+ \) (respectively \( R_t^+ \)) is the spectral function (respectively the resolvent) of \( A^+ \) and \( P^+ \) the operator of projection of \( H^+ \) on \( H \) then the functions operators \( E_t = P^+ E_t^+ \) and \( R_{\lambda} = P^+ R_t^+ \) are said, respectively, generalized spectral function and generalized resolvent of the operator \( A \). They are joined by the relation
\[
(1.8) \quad R_{\lambda} = \int_{\alpha}^{\beta} \frac{dE_t}{t - \lambda}, \quad \lambda \in \rho(A),
\]
in addition, for all real numbers \( \alpha, \beta \) (\( \alpha < \beta \)), we have the Stieltjes inversion formula
\[
(1.9) \quad ([E_\alpha - E_\beta]f, g) = \frac{1}{2\pi i} \lim_{\tau \to \infty} \int_{\alpha}^{\beta} ([R_{\sigma + i\tau} - R_{\sigma - i\tau}]f, g)d\sigma, \quad f, g \in H.
\]
Moreover, for all \( f \) of \( D(A) \):
\[
Af = \int_{-\infty}^{+\infty} t dE_t f.
\]
The generalized spectral function \( E_t \) satisfy the same conditions (ii) and (iii) of \( E_t \) but the first is replaced by
\[(i') \quad E_{t_2} - E_{t_1}, \text{ where } t_2 > t_1, \text{ is a bounded positive operator.}\]
The restriction \( P^+ A^+ \) is said quasi selfadjoint extension of the operator \( A \). It is from this notion that Straus [21] developed his theory of generalized resolvent of a symmetric operator. Let’s designate by \( F_{\lambda} \) the class of all quasi selfadjoint...
linear operators defined on $\mathcal{N}_\lambda$ and that apply $\mathcal{N}_\lambda$ to $\mathcal{N}_{\bar{\lambda}}$. The set of generalized resolvents is defined by

$$\begin{align*}
R_\lambda &= (A_{F(\lambda)} - \lambda I)^{-1}, \\
R_{\bar{\lambda}} &= R^*_\lambda,
\end{align*}$$

where $\lambda_\circ$ is a non real point, $F(\lambda)$ an analytic function operator in the half plane $(\Im m \lambda \Im m \lambda_\circ > 0)$ to value in $\mathcal{F}_{\lambda_\circ}$ and $A_{F(\lambda)}$ $(\Im m \lambda \Im m \lambda_\circ > 0)$ a quasi selfadjoint extension of the operator $A$ defined by

$$D(A_{F(\lambda)}) = D(A) \oplus [F(\lambda) - I] \mathcal{N}_{\lambda_\circ},$$

$$A_{F(\lambda)}(f + F(\lambda) \varphi - \varphi) = Af + \lambda_\circ F(\lambda) \varphi - \bar{\lambda}_\circ \varphi$$

with $f \in D(A)$ and $\varphi \in \mathcal{N}_{\lambda_\circ}$. The adjoint operator $A^*_{F(\lambda)}$ is defined by

$$D(A^*_{F(\lambda)}) = D(A) \oplus [F^*(\lambda) - I] \mathcal{N}_{\bar{\lambda}_\circ},$$

$$A^*_{F(\lambda)}(f + F^*(\lambda) \psi - \psi) = Af + \bar{\lambda}_\circ F^*(\lambda) \psi - \bar{\lambda}_\circ \psi$$

with $f \in D(A)$ and $\psi \in \mathcal{N}_{\bar{\lambda}_\circ}$.

1.4. **Some convergences.** We call $t$ a continuity point of $E_t$ if $E_{t+0} - E_t = 0$.

We call [1] convergence in the mean the convergence in the space $L^2(X, \mu)$ and we denote by

$$f(x) = \text{l.i.m.} f_n(x),$$

if

$$\lim_{n \to \infty} \int_X |f(x) - f_n(x)|^2 \, dx = 0, \quad \text{almost everywhere in } X.$$

($\text{l.i.m.}$ is an abbreviation for limes in medio, i.e. limit in the mean).

2. **Carleman operators**

One can find necessary information about Carleman operators, for example, in [5, 9, 22, 23, 24]. In this section we shall present only part of it. Let $X$ be an arbitrary set, $\mu$ a $\sigma$-finit measure on $X$ ($\mu$ is defined on a $\sigma$-algebra of subsets of $X$, we don’t indicate this $\sigma$-algebra), $L_2(X, \mu)$ the Hilbert space of square integrable functions with respect to $\mu$. Instead of writing ‘$\mu$-measurable’, ‘$\mu$-almost everywhere’ and ‘$(d\mu(x))$’ we write ‘measurable’, ‘a.e’ and ‘$dx$’.

**Definition 1** ([24]). A linear operator $A: D(A) \to L_2(X, \mu)$, where the domain $D(A)$ is a dense linear manifold in $L_2(X, \mu)$, is said to be **integral** if there exists a measurable function $K$ on $X \times X$, a kernel, such that, for every $f \in D(A)$,

$$Af(x) = \int_X K(x, y) f(y) \, dy \quad \text{a.e.}$$
A kernel $K$ on $X \times X$ is said to be Carleman if $K(x, y) \in L_2(X, \mu)$ for almost every fixed $x$, that is to say

$$\int_X |K(x, y)|^2 \, dy < \infty \quad \text{a.e.}$$

(2.2)

An integral operator $A$ with a kernel $K$ is called a Carleman operator if $K$ is a Carleman kernel. Every Carleman kernel $K$ defines a Carleman function $k$ from $X$ to $L_2(X, \mu)$ by $k(x) = K(x, \cdot)$ for all $x$ in $X$ for which $K(x, \cdot) \in L_2(X, \mu)$.

Self-adjoint Carleman operators have generalized eigenfunction expansions, which can be used in the study of linear elliptic operators, see [14]. A general reference for Carleman operators on $L_2$-spaces is [8]. The notion of a Carleman operator has been extended in many directions. By replacing $L_2$ by an arbitrary Banach function space one obtains the so-called generalized Carleman operators (see [18]) and by considering Bochner integrals and abstract Banach spaces one is lead to the so-called Carleman and Korotkov operators on a Banach space ([7]).

Now we consider the class of integral operators (2.1) that we have studied here generated by the following symmetric Carleman kernel

$$K(x, y) = \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)},$$

(2.3)

where the overbar denotes complex conjugation. $\{\psi_p(x)\}_{p=0}^{\infty}$ is an orthonormal sequence in $L^2(X, \mu)$ such that

$$\sum_{p=0}^{\infty} |\psi_p(x)|^2 < \infty \quad \text{a.e.,}$$

(2.4)

and $\{a_p\}_{p=0}^{\infty}$ a real number sequence verifying

$$\sum_{p=0}^{\infty} a_p^2 |\psi_p(x)|^2 < \infty \quad \text{a.e.}$$

(2.5)

We called $\{\psi_p(x)\}_{p=0}^{\infty}$ a Carleman sequence. Let $L(\psi)$ be the closed set of linear combinations of elements of the orthogonal sequence $\{\psi_p(x)\}_{p=0}^{\infty}$. It is lucid that the orthogonal complement $L^\perp(\psi) = L_2(X, \mu) \ominus L(\psi)$ is contained in $D(A)$ and annihilates the operator $A$.

The following lemma [3] tells us when the Carleman operator $A$ possesses equal deficiency indices.

**Lemma 1** ([3]). The operator $A$ possesses equal deficiency indices $n_+(A) = n_-(A) = m$, $(m < \infty)$, if and only if there exist sequences $\{\gamma_p^{(k)}\}_{p=0}^{\infty}$, $(k = 1, 2, \ldots, m)$, verifying
1) For all \( k \)
\[
(2.6) \quad \theta_k(x) = \sum_{p=0}^{\infty} \gamma_p^{(k)} \psi_p(x) \in L^\perp(\psi) \quad (k = 1, 2, \ldots, m)
\]

2) For all \( \lambda \) (\( \Im m \lambda \neq 0 \))
\[
(2.7) \quad \sum_{p=0}^{\infty} \left| \frac{\gamma_p^{(k)}}{a_p - \lambda} \right|^2 < \infty, \quad (k = 1, 2, \ldots, m)
\]

3) The linear space of the sequences \( \{ \gamma_p^{(k)} \}_{p=0}^{\infty} \), \( k = 1, 2, \ldots, m \), verifying 1) and 2) is \( m \) dimension.

3. Generalized resolvents

We first prove the following important lemma.

**Lemma 2.** Let \( B \) be a closed symmetric operator, \( \psi \) the eigenvector of \( B \) belonging to the eigenvalue \( b \). Then \( \psi \in D(B) \) if and only if for a certain \( \lambda \) (\( \Im m \lambda \neq 0 \)) and for all \( \varphi_\lambda \) and \( \tilde{\varphi}_\lambda \)
\[
(\varphi_\lambda, \psi) = (\tilde{\varphi}_\lambda, \psi) = 0,
\]
where \( \varphi_\lambda \) and \( \tilde{\varphi}_\lambda \) belong respectively to the defect spaces \( \mathcal{N}_\lambda \) and \( \mathcal{N}_{\bar{\lambda}} \).

**Proof.** Let \( \psi \in D(B) \) and \( \varphi_\lambda \in \mathcal{N}_\lambda \) (\( \Im m \lambda \neq 0 \)), then
\[
(b \psi, \varphi_\lambda) = (B \psi, \varphi_\lambda) = (\psi, B^* \varphi_\lambda) = \bar{\lambda} (\psi, \varphi_\lambda).
\]
Therefore,
\[
(b - \bar{\lambda}) (\psi, \varphi_\lambda) = 0
\]
and as \( b - \bar{\lambda} \neq 0 \), it follows that \( (\psi, \varphi_\lambda) = 0 \). Now let \( h \) be an arbitrary element of \( D(B^*) \). By the hilbertienne decomposition we have
\[
h = f + \alpha \varphi_\lambda + \beta \tilde{\varphi}_\lambda,
\]
with \( f \in D(B) \), \( \varphi_\lambda \in \mathcal{N}_\lambda \), \( \tilde{\varphi}_\lambda \in \mathcal{N}_{\bar{\lambda}} \), and \( \alpha, \beta \) two complex numbers. Then,
\[
(B^* h, \psi) = (B f, \psi) = (f, b \psi) = (h, b \psi),
\]
that is to say \( \psi \in D(B) \). \( \square \)

Now we suppose that the symmetric Carleman operator \( A \) (2.1) — (2.3) possesses equal deficiency indices \( n_+(A) = n_-(A) = 1 \). By Lemma 1 there exist a sequence \( \{ \gamma_p \}_{p=0}^{\infty} \) such that:
\[
\sum_{p=0}^{\infty} |\gamma_p|^2 = \infty
\]
and verifying the three conditions of the quoted lemma. By (2.6) and (2.7) we conclude that the function
\[
(3.1) \quad \varphi_\lambda(x) = \sum_{p=0}^{\infty} \frac{\gamma_p}{a_p - \lambda} \psi_p(x)
\]
belongs to the defect space $\mathcal{N}_\lambda$ of the operator $A$. In what follows, to facilitate the writing, we will designate by $\hat{A}$ the restriction of $A$ on the subspace $L(\psi)$.

Now we consider the following integral equation

\[(3.2) \quad \int_X \sum_{p=0}^{\infty} a_p \psi_p(x) \overline{\psi_p(y)} Y(y) \, dy - \lambda Y(x) = f(x).\]

Let $f(x) = \sum_{p=0}^{\infty} c_p \psi_p(x) \left( \sum_{p=0}^{\infty} |c_p|^2 < \infty \right)$, then the solution of the equation (3.2) will be the function

\[(3.3) \quad Y(x, \lambda) = \sum_{p=0}^{\infty} \frac{c_p}{a_p - \lambda} \psi_p(x).\]

Let’s notice that the formula (3.3) gives the resolvent of the self-adjoint extension $\hat{A}$ of the operator $\hat{A}$ which possesses a complete system of eigenfunctions $\{\psi_k(x)\}$ of the space $L(\psi)$. The resolvent $\hat{R}_\lambda$ of the operator $\hat{A}$ is an integral operator defined on the space $L(\psi)$:

\[(3.4) \quad \hat{R}_\lambda f = \int_X \hat{K}(x, y; \lambda) f(y) \, dy,\]

where

\[\hat{K}(x, y; \lambda) = \sum_{p=0}^{\infty} \frac{1}{a_p - \lambda} \psi_p(x) \overline{\psi_p(y)}.\]

Any solution of the equation (3.2) in $D(A^*)$ admits the following representation

\[(3.5) \quad Y(x, \lambda) = \hat{R}_\lambda f(x) + c \varphi_\lambda(x),\]

where $c$ is an any complex number.

Let’s put $\lambda_0 = i$, then $F(\lambda)$ (subsection 1.3) can be given by the formula

\[F(\lambda) \varphi_{-i} = \omega(\lambda) \varphi_i\]

with $\omega(\lambda)$ an analytic function in the upper half plane and $|\omega(\lambda)| \leq 1$.

The operator $A_{F(\lambda)}$ is defined on the set $D\left(A_{F(\lambda)}\right)$ as

\[(3.6) \quad \begin{cases} f = x + \omega(\lambda) \varphi_i - \varphi_{-i} (x \in D(A)) , \\ A_{F(\lambda)} f = Ax + i\omega(\lambda) \varphi_i + \varphi_{-i}, \end{cases}\]

then

\[(3.7) \quad D\left(A_{F(\lambda)}\right) = \{ g \in L(\psi) : g = f + [\omega(\lambda) \varphi_i - \varphi_{-i}] c, \; f \in D(A) \}, \]

\[D\left(A^*_{F(\lambda)}\right) = \{ h \in L(\psi) : g = f + [\overline{\omega(\lambda)} \varphi_{-i} - \varphi_i] c, \; f \in D(A) \}.\]

We introduce the following function

\[\nu_\lambda = \overline{\omega(\lambda)} \varphi_{-i} - \varphi_i,\]

then $D\left(A_{F(\lambda)}\right)$ is defined as the set of $y \in D\left(A^*\right)$ for which

\[\langle A^* y, \nu_\lambda \rangle = \langle y, A^*_\lambda \nu_\lambda \rangle.\]
While choosing in (3.5) for all \( \lambda (\Im m \lambda > 0) \) \( c = C(\lambda) \), as we have the equality

\[
(A^* Y, \nu_\lambda) = (Y, A^* \nu_\lambda),
\]

we get a formula giving the set of generalized resolvents in terms of analytic functions \( \omega(\lambda) \). By (3.8) we have

\[
(3.9) \quad C(\lambda) = \frac{[1 - \omega(\lambda)] (f, \varphi_\lambda)}{[\omega(\lambda) \chi(\lambda) - 1] (\lambda + i) (\varphi_\lambda, \varphi_i)} \quad (\Im m \lambda > 0),
\]

where

\[
(3.10) \quad \chi(\lambda) = \frac{\lambda - i (\varphi_\lambda, \varphi_{-i})}{\lambda + i (\varphi_\lambda, \varphi_i)}
\]

denote the characteristic function \([1]\) of operator \( A \). If we substitute (3.9) in (3.5), we get the formula of generalized resolvents

\[
(3.11) \quad R_\lambda f = R_\lambda f + \frac{1 - \omega(\lambda)}{\omega(\lambda) \chi(\lambda) - 1} \frac{(f, \varphi_\lambda)}{(\lambda + i) (\varphi_\lambda, \varphi_i)} \varphi_\lambda \quad (\Im m \lambda > 0).
\]

While taking account that \( R_\lambda = R_\lambda^* \), it is easy to have

\[
(3.12) \quad R_\lambda f = R_\lambda f + \frac{1 - \omega(\lambda)}{\omega(\lambda) \chi(\lambda) - 1} \frac{(f, \varphi_\lambda)}{(\lambda - i) (\varphi_\lambda, \varphi_{-i})} \varphi_\lambda \quad (\Im m \lambda > 0).
\]

So we have demonstrated

**Theorem 1.** Formulas (3.11) and (3.12) establish a bijective correspondence between the set of generalized resolvents of the operator \( A \) and the set of the analytic functions \( \omega(\lambda) \) as \( |\omega(\lambda)| \leq 1 \) \((\Im m \lambda > 0)\). These formulas define the resolvent of a selfadjoint extension of \( A \) in the space \( L(\psi) \) if and only if, \( \omega(\lambda) = \kappa(\text{constant}), \) \(|\kappa| = 1\).

### 4. Generalized spectral functions

Let’s consider the function \( \chi(\lambda) \) given by the formula (3.10):

\[
\chi(\lambda) = \frac{\lambda - i}{\lambda + i} \sum_{p=0}^{\infty} \frac{\gamma_p^2}{(a_p - \lambda)(a_{p+1})},
\]

it’s an analytic function in the half plane \( \Pi = \{ \lambda \in \mathbb{C} : \Im m \lambda \geq 0 \} \) and take its values on the unit disk \( D = \{ \zeta \in \mathbb{C} : |\zeta| \leq 1 \} \), so that the real axis \( \mathbb{R} \) turns into the unit circle \( C = \{ \zeta \in \mathbb{C} : |\zeta| = 1 \} \). Thus, for all \( p = 0, 1, 2, \ldots \), \( \chi(a_p) = 1 \).

Let’s put

\[
\zeta = \frac{\lambda - i}{\lambda + i}.
\]

We can write \([1]\) \( \chi(\lambda) \) under the form

\[
\chi(\lambda) = \chi\left(i \frac{1 + \zeta}{1 - \zeta}\right) = \omega(\zeta) = \frac{(U - \zeta I)^{-1} \varphi_i, \varphi_i)}{(U - \zeta I)^{-1} \varphi_i, \varphi_i)} = \frac{\Phi(\zeta) - ||\varphi_i||^2}{\Phi(\zeta) + ||\varphi_i||^2},
\]
where
\[ \hat{U} = (\hat{A} - iI)(\hat{A} + iI^{-1}) \]
is the unitary Cayley transform of the self-adjoint operator \( \hat{A} \) and
\[ \Phi(\zeta) = \int_0^{2\pi} \frac{e^{is} + \zeta}{e^{is} - \zeta} d(\hat{E}_s \varphi_i, \varphi_i), \]
\( \hat{E}_s \) being the resolution of the identity of the unitary operator \( \hat{U} \). For \( |\zeta| = 1 \), we have
\[ (4.1) \quad \Re e [\Phi(\zeta)] = 0. \]

From the equality
\[ (4.2) \quad (\varphi_\lambda, \varphi_i) = \frac{i}{\lambda + i} [\Phi(\zeta) + \|\varphi_i\|^2] \]
we conclude that
\[ (\varphi_\lambda, \varphi_i) \neq 0 \quad \forall \lambda, \quad \Im \lambda \geq 0. \]

Formulas (4.1) and (4.2) imply that
\[ (4.3) \quad \Im m[(\sigma + i)(\varphi_\sigma, \varphi_i)] = \|\varphi_i\|^2 \quad (\Im m\sigma = 0). \]

Now, we introduce the following useful lemmas:

**Lemma 3.** For all \( f, g \in H \), the functions \( \hat{R}_\lambda f, g \), \( (\varphi_\lambda, \varphi_i) \), \( f, \varphi_\lambda \) and \( (\varphi_\lambda, g) \) are regular on all the complex plane except to points \( a_p \) \( (p = 0, 1, 2, \ldots) \), where they admit simple poles. Besides, the following equalities are true:
\[ \text{res}_{\lambda=a_p} \hat{R}_\lambda f, g = \text{res}_{\lambda=a_p} \frac{(f, \varphi_\lambda)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = (f, \psi_p)(\psi_p, g), \]
\[ \text{res}_{\lambda=a_p} \frac{(f, \varphi_\lambda)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = \text{res}_{\lambda=a_p} \frac{(f, \varphi_\lambda)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = (f, \psi_p)(\psi_p, g). \]

**Proof.** The fact that the mentioned functions are regular on the complex plane except to poles \( a_p \) \( (p = 0, 1, 2, \ldots) \) result from formulas (3.1) and
\[ (\hat{R}_\lambda f, g) = \sum_{p=0}^{\infty} \frac{(f, \psi_p)(\psi_p, g)}{a_p - \lambda}. \]

Furthermore we have:
\[ \text{res}_{\lambda=a_p} \hat{R}_\lambda f, g = (f, \psi_p)(\psi_p, g), \]
it is easy to see that the function
\[ \frac{(f, \varphi_\lambda)(\varphi_\lambda, g)}{(\lambda - i)(\varphi_\lambda, \varphi_i)} = \frac{[\sum_{p=0}^{\infty} \frac{\gamma_p(f, \psi_p)}{a_p - \lambda}] [\sum_{p=0}^{\infty} \frac{\gamma_p(\psi_p, g)}{a_p - \lambda}]}{(\lambda - i) \sum_{p=0}^{\infty} \frac{\gamma^2_p}{(a_p - \lambda)(a_p - i)}}, \]
admits the same residue to the point \( \lambda = a_p \).

The second equality can be verified in the same way.
Lemma 4 ([21]). Let $\varphi (\lambda)$ an analytic function in the half-plane $\Pi^+$ with a positive imaginary part and $\psi (\lambda)$ an analytic function in a certain domain containing the interval $[\alpha, \beta]$. Then we have the formula

$$
\frac{1}{2\pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ \varphi (\lambda) \psi (\lambda) - \varphi (\lambda) \psi (\lambda) \right] d\sigma = \int_{\alpha}^{\beta} \psi (\sigma) d\rho (\sigma) \quad (\lambda = \sigma + i\tau),
$$

with

$$
\rho (\sigma) = \frac{1}{\pi} \lim_{\tau \to +0} \int_{0}^{\sigma} \Im \varphi (t + i\tau) \, dt.
$$

Let $\omega (\lambda)$ be an arbitrary analytic function who applies the half-plane $\Pi^+$ on the unit disk $D$. It is known that the spectral function $E_t$ is uniform and that we can get it by the formula of Stieltjes (1.9):

for all $f (s)$ and $g (s)$ of $L$ and for all reals $\alpha$ and $\beta$ $(\alpha < \beta)$ we have the equality

$$(E_{\alpha,\beta} f, g) = \frac{1}{2\pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ \mathcal{R}_{\re + i\tau} - \mathcal{R}_{\re - i\tau} \right] f, g \right) d\sigma$$

with

$$E_{\alpha,\beta} = (E_{\beta} + E_{\beta+0}) / 2 - (E_{\alpha} + E_{\alpha+0}) / 2.$$
Then \((\lambda = \sigma + i\tau)\)

\[
\frac{1}{2\pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} \left[ f_1(\lambda) - f_2(\lambda) \right] d\sigma = \frac{1}{2\pi i} \int_{\alpha}^{\beta} \frac{2i\Im[(\sigma + i)(\varphi_\sigma, \varphi_i)](f, \varphi_\sigma)}{(\sigma^2 + 1)|\varphi_\sigma, \varphi_i|^2} d\sigma \\
+ \sum_{\alpha_k \in (\alpha, \beta)} c_k \psi_k(s),
\]
c_k being coefficients in the development (4.5).

Now, we notice that for all analytic function \(\omega(\lambda)\) in the half-plane \(\Pi^+\) as \(|\omega(\lambda)| \leq 1\), we obtain

\[
\Im i\omega(\lambda)\chi(\lambda) - 1 > 0 \quad (\Im \lambda > 0).
\]

After this, while using the Lemma 2 and the equality (4.3), we get

(4.6) \(E_{\alpha, \beta} f = \frac{1}{2\pi i} \lim_{\tau \to +0} \int_{\alpha}^{\beta} [R_\lambda - R_\beta] f d\sigma = \int_{\alpha}^{\beta} \frac{(f, \varphi_\sigma) \varphi_\sigma}{(\sigma^2 + 1)|\varphi_\sigma, \varphi_i|^2} d\rho(\sigma),\)

with

(4.7) \(\rho(\sigma) = \frac{1}{\pi} \lim_{\tau \to +0} \int_{0}^{\sigma} \left[ \Im m \frac{-2i}{\omega(\lambda)\chi(\lambda) - 1} \right] dt, \quad (\lambda = t + i\tau)\)

and

\[\tilde{\varphi}_i(s) = \frac{\varphi_i(s)}{||\varphi_i||}.\]

The function \(\rho(\sigma)\) is decreasing because

\[
\Re e \frac{1}{\omega(\lambda)\chi(\lambda) - 1} \geq \frac{1}{1 + |\omega(\lambda)\chi(\lambda)|} \geq \frac{1}{2}.
\]

Thus, we have demonstrated the theorem

**Theorem 2.** Let \(\omega(\lambda)\) be an analytic function in the half-plane \(\Pi^+\) and \(E_t (\omega)\) \((-\infty < t < +\infty)\) the spectral function of the operator \(A\). Then for all \(f(s)\) of \(L(\psi)\) and for all reals \(\alpha\) and \(\beta\) \((\alpha < \beta)\) we have the relation (4.6) and the following equalities

\[
(E_{\alpha, \beta} f, f) = \int_{\alpha}^{\beta} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)|\varphi_\sigma, \varphi_i|^2} d\rho(\sigma),
\]

\[
f(s) = \frac{l.i.m.}{\alpha \to -\infty, \beta \to +\infty} \int_{\alpha}^{\beta} \frac{(f, \varphi_\sigma) \varphi_\sigma(s)}{(\sigma^2 + 1)|\varphi_\sigma, \varphi_i|^2} d\rho(\sigma),
\]

\[
(f, f) = \int_{-\infty}^{+\infty} \frac{|(f, \varphi_\sigma)|^2}{(\sigma^2 + 1)|\varphi_\sigma, \varphi_i|^2} d\rho(\sigma),
\]

where \(\rho(\sigma)\) is defined by the formula (4.7) for \(\lambda = \sigma + i\tau, \Im \lambda > 0.\)
Corollary 1. In order that \( t (\infty < t < +\infty) \) be a continuity point of the spectral function \( E_t \) of the operator \( A \) it is necessary and sufficient that it is a continuity point of the function \( \rho (\sigma) \).

Let's consider the formula (4.7). The function \( \chi (\lambda) \) applies all interval \( (a_{p_k}, a_{p_{k+1}}) \) (we suppose that \( a_{p_k} \) and \( a_{p_{k+1}} \) are neighboring points) in the unit disk. The homographic transform \( \frac{1+z}{1-z} \) applies the circle \( |z| = r \leq 1 \) in the not euclidean circle of center \( i \) such that the image of \( r = 0 \) will be the point \( i \) and the image of \( r = 1 \) will be the real axis \( \mathbb{R} \). Therefore, for \( \omega (\lambda) = 1, \rho (\sigma) \) is a jumps function with points jumps \( a_{p_k} \) and for \( \omega (\lambda) = \kappa (\kappa = \text{constant with } |\kappa| < 1) \), \( \rho (\sigma) \) is absolutely continuous.

With the help of the self-adjoints extensions \( (\omega (\lambda) = \kappa = \exp(i\varphi)) \rho (\sigma) \) will be a jumps function with points jumps \( \sigma_p \) for whom \( \chi (\sigma_p) = \exp(-i\varphi) \).

Of the pace of the function \( \rho (\sigma) \) we are convinced of the following findings.

Corollary 2. The quasi-self-adjoint extension associated to the analytical function \( \omega (\lambda) \) (\(|\omega (\lambda)| \leq 1 \text{ in } \Pi^+ \text{ and } |\omega (\sigma)| = 1 \text{ for } \Im \sigma = 0 \)) admits a merely point spectrum.

Corollary 3. The interval \((c,d) (\infty \leq c < d \leq +\infty)\) doesn't contain the spectrum points of the self-adjoint extension of the operator \( A \) generated by the functions \( \omega (\lambda) \) if and only if \( \omega (\lambda) \) verify the following conditions:

a) \( \omega (\lambda) \) is analytic in \( \Pi^+ \) and \(|\omega (\lambda)| \leq 1 (\Im \lambda > 0)\);

b) \( \omega (\lambda) \) admits an extension by continuity from \( \Pi^+ \) on \((c,d)\);

c) \(|\omega (\sigma)| = 1, \text{ if } \sigma \in (c,d)\);

d) \( \omega (\sigma) \neq \chi (\sigma) \) for \( \sigma \in (c,d) \).

If we suppose in (2.3) that \( a_p > 0 \), then \( A \) will be a positive operator. Thus the Corollary 3 give the criteria to get the positive spectral functions. In particular self-adjoint extension possessed a positive spectral function if it is generated by functions \( \omega (\lambda) = \kappa = \exp (i\varphi), 0 \leq \varphi \leq \varphi_0, \chi (0) = \exp (-i\varphi_0) \).

References


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