ON ENDMORPHISMS OF MULTIPLICATION AND COMULTIPLICATION MODULES

H. Ansari-Toroghy and F. Farshadifar

Abstract. Let $R$ be a ring with an identity (not necessarily commutative) and let $M$ be a left $R$-module. This paper deals with multiplication and comultiplication left $R$-modules $M$ having right $\text{End}_R(M)$-module structures.

1. Introduction

Throughout this paper $R$ will denote a ring with an identity (not necessarily commutative) and all modules are assumed to be left modules. Further “$\subset$” will denote the strict inclusion and $\mathbb{Z}$ will denote the ring of integers.

Let $M$ be a left $R$-module and let $S := \text{End}_R(M)$ be the endomorphism ring of $M$. Then $M$ has a structure as a right $S$-module so that $M$ is an $R-S$ bimodule. If $f: M \to M$ and $g: M \to M$, then $fg: M \to M$ defined by $m(fg) = (mf)g$.

Also for a submodule $N$ of $M$,

$$I^N := \{ f \in S : \text{Im}(f) = Mf \subseteq N \}$$

and

$$I_N := \{ f \in S : N \subseteq \text{Ker}(f) \}$$

are respectively a left and a right ideal of $S$. Further a submodule $N$ of $M$ is called (3) an open (resp. a closed) submodule of $M$ if $N = N^\circ$, where $N^\circ = \sum_{f \in I_N} \text{Im}(f)$ (resp. $N = \bar{N}$, where $\bar{N} = \cap_{f \in I_N} \text{Ker}(f)$). A left $R$-module $M$ is said to self-generated (resp. self-cogenerated) if each submodule of $M$ is open (resp. is closed).

Let $M$ be an $R$-module and let $S = \text{End}_R(M)$. Recently a large body of researches has been done about multiplication left $R$-module having right $S$-module structures. An $R$-module $M$ is said to be a multiplication $R$-module if for every submodule $N$ of $M$ there exists a two-sided ideal $I$ of $R$ such that $N = IM$.

In [2], H. Ansari-Toroghy and F. Farshadifar introduced the concept of a comultiplication $R$-module and proved some results which are dual to those of multiplication $R$-modules. An $R$-module $M$ is said to be a comultiplication $R$-module if for every submodule $N$ of $M$ there exists a two-sided ideal $I$ of $R$ such that $N = (0 :_M I)$.

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This paper deals with multiplication and comultiplication left $R$-modules $M$ having right $\text{End}_R(M)$-modules structures. In section three of this paper, among the other results, we have shown that every comultiplication $R$-module is co-Hopfian and generalized Hopfian. Further if $M$ is a comultiplication module satisfying ascending chain condition on submodules $N$ such that $M/N$ is a comultiplication $R$-module, then $M$ satisfies Fitting’s Lemma. Also it is shown that if $R$ is a commutative ring and $M$ is a multiplication $R$-module and $S$ is a domain, then for every maximal submodule $P$ of $M$, $IP$ is a maximal ideal of $S$.

2. Previous results

In this section we will provide the definitions and results which are necessary in the next section.

**Definition 2.1.**

(a) $M$ is said to be (see [9]) a **multiplication** $R$-module if for any submodule $N$ of $M$ there exists a two-sided ideal $I$ of $R$ such that $N = IM$.

(b) $M$ is said to be a **comultiplication** $R$-module if for any submodule $N$ of $M$ there exists a two-sided ideal $I$ of $R$ such that $N = (0 :_M I)$.

(c) Let $N$ be a non-zero submodule of $M$. Then $N$ is said to be (see [1]) **large or essential** (resp. **small**) if for every non-zero submodule $L$ of $M$, $N \cap L \neq 0$ (resp. $L + N = M$ implies that $L = M$).

(d) $M$ is said to be (see [7]) **Hopfian** (resp. **generalized Hopfian** ($gH$ for short)) if every surjective endomorphism $f$ of $M$ is an isomorphism (resp. has a small kernel).

(e) $M$ is said to be (see [8]) **co-Hopfian** (resp. **weakly co-Hopfian**) if every injective endomorphism $f$ of $M$ is an isomorphism (resp. an essential homomorphism).

(f) An $R$-module $M$ is said to satisfy **Fitting’s Lemma** if for each $f \in \text{End}_R(M)$ there exists an integer $n \geq 1$ such that $M = \text{Ker}(f^n) \oplus \text{Im}(f^n)$ (see [5]).

(g) Let $M$ be an $R$-module and let $I$ be an ideal of $R$. Then $IM$ is called to be **idempotent** if $I^2M = IM$.

3. Main results

**Lemma 3.1.** Let $R$ be any ring. Every comultiplication $R$-module is co-Hopfian.

**Proof.** Let $M$ be a comultiplication $R$-module and let $f : M \rightarrow M$ be a monomorphism. There exists a two-sided ideal $I$ of $R$ such that $\text{Im}(f) = (0 :_M I)$. Now let $m \in M$ so that $mf \in \text{Im}(f)$. Then for each $a \in I$, we have $(am)f = a(mf) = 0$. It follows that $am \in \text{Ker}(f) = 0$. This implies that $am = 0$ so that $m \in (0 :_M I) = Mf$. Hence we have $M \subseteq Mf$ so that $f$ is epic. It follows that $M$ is a co-Hopfian $R$-module. \[\square\]
The following examples shows that not every comultiplication (resp. Artinian) \( R \)-module is an Artinian (resp. a comultiplication) \( R \)-module.

**Example 3.2.** Let \( p \) be a prime number. Then let \( R \) be the ring with underlying group
\[
R = \text{End}_\mathbb{Z} \left( \mathbb{Z}(p^\infty) \right) \oplus \mathbb{Z}(p^\infty),
\]
and with multiplication
\[
(n_1, q_1) \cdot (n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1).
\]
Osofsky has shown that \( R \) is a non-Artinian injective cogenerator (see [6, Exa. 24.34.1]). In fact \( R \) is a commutative ring. Hence \( R \) is a comultiplication \( R \)-module by [6, Prop. 23.13].

**Example 3.3.** Let \( F \) be a field, and let \( M = \bigoplus_{i=1}^{n} F_i \), where \( F_i = F \) for \( i = 1, 2, \ldots, n \). Clearly \( M \) is an Artinian non-comultiplication \( F \)-module.

**Theorem 3.4.** Let \( M \) be a comultiplication module satisfying ascending chain condition on submodules \( N \) such that \( M/N \) is a comultiplication \( R \)-module. Then \( M \) satisfies Fitting’s Lemma.

**Proof.** Let \( f \in \text{End}_R(M) \) and consider the sequence
\[
\text{Ker} f \subseteq \text{Ker} f^2 \subseteq \cdots.
\]
Since every submodule of a comultiplication \( R \)-module is a comultiplication \( R \)-module by [2], for each \( n \) we have \( M/\text{Ker} f^n \cong \text{Im} f^n \) implies that \( M/\text{Ker} f^n \) is a comultiplication \( R \)-module. Hence by hypothesis there exists a positive integer \( n \) such that \( \text{Ker}(f^n) = \text{Ker}(f^{n+h}) \) for all \( h \geq 1 \). Set \( f^n_1 = f^n |_{M(f^n)} \). Then \( f^n_1 \in \text{End}_R(M(f^n)) \). Further we will show that \( f^n_1 \) is monic. To see this let \( x \in \text{Ker}(f^n_1) \). Then \( x = y(f^n) \) for some \( y \in M \) and we have \( x(f^n) = 0 \). It follows that \( y(f^{2n}) = 0 \) so that
\[
y \in \text{Ker}(f^{2n}) = \text{Ker}(f^n).
\]
Hence we have \( x = 0 \). But \( M f^n \) is a comultiplication \( R \)-module and every comultiplication \( R \)-module is co-Hopfian by Lemma 3.1. So we conclude that \( f^n_1 \) is an automorphism. In particular, \( M(f^n) \cap \text{Ker}(f^n) = 0 \). Now let \( x \in M \). Since \( f^n_1 \) is epimorphism, then there exists \( y \in M \) such that \( x(f^n) = y(f^{2n}) \). Hence \( (x - y(f^n))(f^n) = 0 \). It follows that \( x - y(f^n) \in \text{Ker}(f^n) \). Now the result follows from this because \( x = y(f^n) + (x - y(f^n)) \). \( \square \)

**Corollary 3.5.** Let \( M \) be an indecomposable comultiplication module satisfying ascending chain condition on submodules \( N \) such that \( M/N \) is a comultiplication \( R \)-module. Let \( f \in \text{End}_R(M) \). Then the following are equivalent.

(i) \( f \) is a monomorphism.
(ii) \( f \) is an epimorphism.
(iii) \( f \) is an automorphism.
(iv) \( f \) is not nilpotent.
Proof. (i)⇒(ii). This is clear by Lemma 3.1.
(iii)⇒(ii). This is clear.
(iii)⇒(iv). Assume that \( f \) is an automorphism. Then \( M = Mf \). Hence,
\[
M = Mf = M(f^2) = \cdots.
\]
If \( f \) were nilpotent, then \( M \) would be zero.
(ii)⇒(i). Assume that \( f \) is an epimorphism. Then \( M = Mf \). Hence
\[
M = Mf = M(f^2) = \cdots.
\]
By Theorem 3.4, there is a positive integer \( n \) such that
\[
M = \ker(f^n) \oplus \text{Im}(f^n).
\]
Hence \( M = \ker(f^n) \oplus M \), so \( \ker(f^n) = 0 \). Thus, \( \ker(f) = 0 \).
(ii)⇒(iii). This follows from (ii)⇒(i).
(iv)⇒(iii). Suppose that \( f \) is not nilpotent. By Theorem 3.4, there exists a positive integer \( n \) such that \( M = Mf^n \oplus \ker f^n \). Since \( M \) is indecomposable \( R \)-module, it follows that \( \ker f^n = 0 \) or \( Mf^n = 0 \). Since \( f \) is not nilpotent, we must have \( \ker f^n = 0 \). This implies that \( f \) is monic. This in turn implies that \( f \) is epic by Lemma 3.1. Hence the proof is completed. \( \square \)

Example 3.6. Let \( A = K[x, y] \) be the polynomial ring over a field \( K \) in two indeterminates \( x, y \). Then \( \overline{A} = A/(x^2, y^2) \) is a comultiplication \( \overline{A} \)-module. But \( \overline{A}/\overline{Ax\bar{y}} \) is not a comultiplication \( \overline{A} \)-module (see [6, Exa. 24.4]). Therefore, not every homomorphic image of a comultiplication module is a comultiplication module.

Remark 3.7. In the Corollary 3.5 the condition \( M \) satisfying ascending chain condition on submodules \( N \) such that \( M/N \) is a comultiplication \( R \)-module can not be omitted. For example \( M = \mathbb{Z}(p^\infty) \) is an indecomposable comultiplication \( \mathbb{Z} \)-module but not satisfying ascending chain condition on submodules \( N \) such that \( M/N \) is a comultiplication \( \mathbb{Z} \)-module. Define \( f: \mathbb{Z}(p^\infty) \rightarrow \mathbb{Z}(p^\infty) \) by \( x \rightarrow px \). Clearly \( f \) is an epimorphism with \( \ker f = \mathbb{Z}(1/p + \mathbb{Z}) \). Hence \( f \) is not a monomorphism.

Lemma 3.8. Let \( M \) be a comultiplication \( R \)-module and let \( N \) be an essential submodule of \( M \). If the right ideal \( I_N \) of \( \text{End}_R(M) \) is non-zero, then it is small in \( \text{End}_R(M) \).

Proof. Let \( J \) be any right ideal of \( S = \text{End}_R(M) \) such that \( I_N + J = S \). Then \( 1_M = f + j \) for some \( f \in I_N \) and \( j \in J \). Since \( \ker(1_M - f) \cap N = 0 \) and \( N \) is an essential submodule of \( M \), it follows that \( j \) is a monomorphism. Hence by Lemma 3.1, \( j \) is an automorphism so that \( J = S \). Hence \( I_N \) is a small right ideal of \( S \). \( \square \)

Proposition 3.9. Let \( M \) be a comultiplication \( R \)-module and let \( N \) be a submodule of \( M \) such that \( M/N \) is a faithful \( R \)-module. Then \( M/N \) is a co-Hopfian \( R \)-module.

Proof. Let \( f: M/N \rightarrow M/N \) be an \( R \)-monomorphism and \( (M/N)f = K/N \), with \( N \subseteq K \subseteq M \). Since \( M \) is a comultiplication \( R \)-module there exists a two-sided ideal \( I \) of \( R \) such that \( K = (0 :_M I) \). Now
\[
(I(M/N))f = I(M/N)f = I(K/N) = 0.
\]
Since $f$ is monic, it follows that $I(M/N) = 0$. This in turn implies that $I \subseteq \text{Ann}_R(M/N) = 0$. Hence we have $K = M$ so that $f$ is an epimorphism. \hfill \Box

Lemma 3.10. Every comultiplication $R$-module is $gH$.

Proof. Let $M$ be comultiplication $R$-module and let $f : M \to M$ be an epimorphism and assume that $\ker(f) + K = M$, where $K$ is a submodule of $M$. So $Kf = Mf = M$. Since $M$ is a comultiplication module, there exists a two-sided ideal $J$ of $R$ such that $K = (0 :_M J)$.

Now $0 = 0f = (J(0 :_M J))f = J(Kf) = JM$.

It follows that $J \subseteq \text{Ann}_R(M)$. Hence we have $K = (0 :_M J) = M$. This shows that $\ker(f)$ is a small submodule of $M$. So the proof is completed. \hfill \Box

Proposition 3.11. 

(a) Assume that whenever $f, g \in \text{End}_R(M)$ with $fg = 0$ then we have $gf = 0$. If $M$ is a self-generated (resp. self-cogenerated) $R$-module, then $M$ is Hopfian (resp. co-Hopfian).

(b) Let $M$ be a self-generated (resp. self-cogenerated) $R$-module and let $S$ be a left Noetherian (resp. right Artinian) ring. Then $M$ is a Noetherian $S$-module.

Proof. (a) Let $S = \text{End}_R(M)$ and let $g : M \to M$ be an epimorphism. Let $f$ be any element of $I^{\ker(g)}$. Then $Mf \subseteq \ker(g)$, so $M(fg) = (Mf)g = 0$. Hence, $fg = 0$. By our assumption, $gf = 0$. Since $g$ is an epimorphism, we have

$$Mf = (Mg)f = M(gf) = 0.$$ 

Thus, if $M$ is self-generated,

$$\ker(g) = \sum_{f \in I^{\ker(g)}} \text{Im}(f) = 0.$$ 

Hence $M$ is a Hopfian $R$-module. The proof is similar when $M$ is a self-cogenerated $R$-module.

(b) Let

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

be an ascending chain of $S$-submodules of $M$. This induces the sequence

$$I^{N_1} \subseteq I^{N_2} \subseteq \cdots \subseteq I^{N_k} \subseteq \cdots.$$ 

Now there exists a positive integer $s$ such that for each $0 \leq i$, $I^{N_s} = I^{N_{i+s}}$. Since $M$ is a self-generated $R$-module, we have $N_s = MI^{N_s} = MI^{N_{i+s}} = N_{i+s}$ for every $0 \leq i$. Thus $M$ is a Noetherian $S$-module. For right Artinian case when $M$ is a self-cogenerator $R$-module, the proof is similar. So the proof is completed. \hfill \Box

Theorem 3.12. Let $M$ be a multiplication $R$-module and let $N$ be a submodule of $M$.

(a) If $R$ is a commutative ring, and $I$ is an ideal of $R$ such that $IM$ is an idempotent submodule of $M$, then $IM$ is $gH$. 

(b) If $R$ is a commutative ring and $N$ is faithful, then $N$ is weakly co-Hopfian.

(c) If $M$ is a quasi-injective, $N$ is $gH$.

**Proof.** (a) Let $I$ be an ideal of $R$ such that $IM$ be an idempotent submodule of $M$. Let $f : IM \to IM$ be an epimorphism and assume that $\ker(f) + L = IM$, where $L$ is a submodule of $IM$. Then we have $I(\ker(f)) + IL = IM$. Let $\ker(f) = JM$ for some ideal $J$ of $R$. Since $R$ is a commutative ring, we have

$$0 = I(\ker(f))f = (IJM)f = J(IM)f = JIM = IJM = I(\ker(f)).$$

Thus by the above arguments, $IL = IM$ so that $IM \subseteq L$. It follows that $IM = L$ so that $IM$ is a generalized Hopfian $R$-module.

(b) Let $I$ be an ideal of $R$ such that $N = IM$. Let $f : N \to N$ be an injective homomorphism and assume that $Nf \cap K = 0$, where $K$ is a submodule of $N$. Then there exist ideals $J_1$ and $J_2$ of $R$ such that $Nf = J_1M$ and $K = J_2M$. Then we have

$$0 = K \cap Nf = K \cap (IM)f = (J_2M) \cap (IM)f = J_2M \cap J_1M \supseteq J_2J_1M.$$

Hence $J_2J_1M = 0$. Now we have

$$(IJ_2M)f = J_2(IM)f = J_2J_1M = 0.$$  

Since $f$ is monic, $J_2N = IJ_2M = 0$. Since $N$ is a faithful $R$-module, we have $J_2 = 0$ so that $K = 0$. Hence $Nf$ is essential in $N$. It implies that $N$ is a weakly co-Hopfian $R$-module as desired.

(c) Let $f : N \to N$ be an epimorphism and let $\ker(f) + K = N$, where $K$ is a submodule of $N$. Since $M$ is quasi-injective, we can extend $f$ to $g : M \to M$. But as $M$ is a multiplication module, $Kg \subseteq K$, therefore $Kf \subseteq K$. On the other hand, $Kf = N$ since $f$ is epimorphism. Therefore $K = N$. Hence $N$ is a generalized Hopfian $R$-module as desired.  

**Proposition 3.13.** Let $R$ be a commutative ring and let $M$ be a multiplication $R$-module. Let $S = \text{End}_R(M)$ be a domain. Then the following assertions hold.

(a) Each non-zero element of $S$ is a monomorphism.

(b) If $I$ and $J$ are ideals of $S$ such that $I \neq J$, then $MI \neq MJ$.

**Proof.** (a) Assume that $0 \neq g \in S$. Then there exist ideals $I$ and $J$ of $R$ such that $\text{Im}(g) = JM$ and $\ker(g) = IM$. Now we have

$$0 = (\ker(g))g = (IM)g = I(Mg) = IJM.$$  

It implies that $IJ \subseteq \text{Ann}_R(M)$. Since $S$ is a domain, $\text{Ann}_R(M)$ is a prime ideal of $R$ by [2, 2.3]. Hence $I \subseteq \text{Ann}_R(M)$ or $J \subseteq \text{Ann}_R(M)$ so that $IM = 0$ or $JM = 0$. It turns out that $\ker(g) = 0$ as desired.

(b) Since $R$ is a commutative ring, $M$ is a multiplication $S$-module. Hence for $0 \neq m \in M$ there exists an ideal $K$ of $S$ such that $mS = MK$. Now we assume that $MI = MJ$. Since $R$ is a commutative ring, $S$ is a commutative ring by [3]. Hence

$$mI = mSI = (MK)I = (MI)K = (MJ)K = (MK)J = mSJ = mJ.$$
Choose \( f \in I \setminus J \). Then since \( mf \in mI = mJ \), there exists \( h \in J \) such that \( mh = mf \). Thus we have \( m(h - f) = 0 \). Further \( h - f \neq 0 \). So by using part (a), we have \( m \in \ker(h - f) = 0 \). But this is a contradiction and the proof is completed.

**Corollary 3.14.** Let \( R \) be a commutative ring and \( M \) be a multiplication \( R \)-module. Set \( S = \text{End}_R(M) \) and \( \text{Im}(J) = \sum_{f \in J} \text{Im}(f) \), where \( J \) is an ideal of \( S \). If \( J \) is a proper ideal of a domain \( S \), then \( \text{Im}(J) \) is a proper submodule of \( M \).

**Proof.** This is an immediate consequence of Proposition 3.13 (b).

**Theorem 3.15.** Let \( R \) be a commutative ring and let \( M \) be a multiplication \( R \)-module such that \( S = \text{End}_R(M) \) is a domain. Then for every maximal submodule \( P \) of \( M \), \( I^P \) is a maximal ideal of \( S \).

**Proof.** Since \( \text{Id}_M \in S \) and \( \text{Id}_M \not\in IP \), we have \( IP \neq S \). Now assume that \( U \) is an ideal of \( S \) such that \( IP \subseteq U \subseteq S \). Then if \( MU = M \), then by Corollary 3.14, \( U = S \). If \( MU = P \), then \( U \subseteq IP \), so \( U = IP \). Hence \( IP \) is a maximal ideal of \( S \) and the proof is completed.

**Example 3.16.** Let \( R \) be a commutative ring and let \( P \) be a prime ideal of \( R \). Set \( M = R/P \). Then \( M \) is a multiplication \( R \)-module and \( S = \text{End}_R(M) \) is a domain. Hence by Theorem 3.15, for every maximal submodule \( N \) of \( M \), \( IN \) is a maximal ideal of \( S \).

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