František Neuman
On ordering vertices of infinite trees

Časopis pro pěstování matematiky, Vol. 91 (1966), No. 2, 170--177

Persistent URL: http://dml.cz/dmlcz/108099

Terms of use:

© Institute of Mathematics AS CR, 1966

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
ON ORDERING VERTICES OF INFINITE TREES

FRANTIŠEK NEUMAN, Brno

(Received December 30, 1964)

1. By a finite (infinite) tree we understand a non-oriented connected graph without circles and with finitely (denumerably) many vertices. A tree is either a finite or an infinite tree. Denote by \{T\} the set of the vertices of a tree \(T\). The edge of the tree \(T\) connecting vertices \(a\) and \(b\) will be denoted by \((a, b)\). A finite path of a tree \(T\) is a finite sequence of its edges \((z_i, z_{i+1}), i = 0, 1, 2, \ldots, n - 1\), such that \(z_i \neq z_{i+1}\) for \(i_1 \neq i_2\). Such a path is said to be of length \(n\), it starts with the vertex \(z_0\) and ends in \(z_n\), and may be denoted by \((z_0, \ldots, z_n)\). The vertices \(z_0\) and \(z_n\) are called the end-vertices of this path, the remaining vertices are termed inner. An infinite path of a tree \(T\) is an infinite sequence of its edges \((z_i, z_{i+1}), i = 0, 1, 2, \ldots\), such that \(z_i \neq z_{i+1}\) for \(i_1 \neq i_2\). We say that this path begins in the vertex \(z_0\) or that \(z_0\) is its end-vertex, the other vertices are its inner vertices. Furthermore, a vertex \(x\) of a tree \(T\) is said to be between the vertices \(a\) and \(b\) of the same tree if there exists a finite path \((a, \ldots, b)\) of \(T\) such that \(x\) is its inner vertex. The distance \(d(a, b)\) of distinct vertices \(a, b\) of a tree \(T\) is the length of the path \((a, \ldots, b)\); and for \(x \in \{T\}\) set \(d(x, x) = 0\).

If \(\tau_i\) denotes a finite sequence \(t_1, t_2, \ldots, t_n\), then let \(\tau_1, \tau_2, \tau_3, \ldots\) denote the sequence \(t_1^1, t_2^1, t_3^1, \ldots, t_n^1, t_1^2, t_2^2, t_3^2, \ldots, t_n^2, t_1^3, t_2^3, \ldots, t_n^3, \ldots\).

Let \(T\) be a finite tree, \(a, b \in \{T\}, a \neq b\). Then a \(2-(a, b)\)-ordering of \(T\) is an ordering of the set \(\{T\}\) into a simple finite sequence beginning with \(a\), ending with \(b\) and such that the distance of consecutive members of this sequence is at most 2 (in the metric of \(T\)). If there exists a \(2-(a, b)\)-ordering of \(T\), we shall say that \(T\) can be \(2-(a, b)\)-ordered; and otherwise that \(T\) cannot be \(2-(a, b)\)-ordered. Furthermore, we shall say that it is possible to \(2\)-order a finite tree \(T\) if there exist \(x, y \in \{T\}, x \neq y\), such that \(T\) can be \(2-(x, y)\)-ordered. Analogously, consider a tree \(U\) (finite or infinite), and an \(a \in \{U\}\). A \(2-a\)-ordering of the tree \(U\) is an ordering of the set \(\{U\}\) into a simple sequence (finite or infinite), starting with \(a\) and such that the distance of consecutive members of this sequence is at most 2 in the metric of \(U\). If there exists a \(2-a\)-ordering of the tree \(U\), we shall say that \(U\) can be \(2-a\)-ordered.

In [2] there were obtained necessary and sufficient conditions under which it is

1) For distinct vertices \(a, b\) of a tree \(T\) the path \((a, \ldots, b)\) is uniquely determined [1, p. 165, theorem 1 (6)].
possible to 2-(a, b)-order or 2-order a finite tree. M. Sekanina dealt with similar orderings of the vertices of a graph, [3], [4]. In this paper we shall give a necessary and sufficient condition under which it is possible to 2-a-order an infinite tree. This problem is due to G. A. Dirac.

2. We shall recall the results of [2] needed in the sequel.

Theorem. Let T be a finite tree, a, b two distinct vertices. It is possible to 2-(a, b)-order T if and only if for the subtree T_1 obtained from T by omitting all vertices of order 1 and all edges incident to these except for a, b there holds:

1° the degree of all vertices in T_1 is at most 4,
2° all vertices of degree 3 and 4 in T_1 are inner vertices of the path (a, ..., b),
3° between every two vertices of degree 4 in T_1 there exists at least one vertex of degree 2 in T. If the degree of a is greater than 1 in T, then between it and the nearest vertex of degree 4 in T_1 there exists a vertex of degree 2 in T. Similarly for b. If the degree of both vertices a and b is greater than 1 in T, then there exists between them at least one vertex of degree 2 in T.

Corollary. A finite tree T can be 2-ordered if and only if for the subtree T_2 (the empty tree and the tree consisting of one vertex are now permitted) obtained from T by omitting all vertices of order 1 and incident edges there holds:

1° the degree of all vertices in T_2 is at most 4,
2° in T_2 there exists a path containing all vertices of degree 3 and 4 (in T_2),
3° between every two vertices of degree 4 in T_2 there exists at least one vertex of degree 2 in T.

3. Let T_0 be an infinite tree and let all its vertices be of a finite degree, a_0 ∈ {T_0}. Then there exists an infinite path of T_0 which begins with a_0.

When we omit a_0 and all incident edges from T_0, we obtain a finite number of components, at least one of which is infinite. Denote one of these infinite components by T_1, an infinite tree. Let a_1 ∈ {T_1}, μ(a_0, a_1) = 1, where μ is the metric in T_0. In general, after omitting the vertex a_j and all incident edges from an infinite tree T_j, we obtain a finite number of components, at least one of which is infinite. Denote by T_{j+1} one of these infinite components. Let a_{j+1} ∈ {T_{j+1}}, μ(a_j, a_{j+1}) = 1 (in T_0). Evidently a_0, a_1, a_2, a_3, ... and (a_0, a_1), (a_1, a_2), (a_2, a_3), ... are the vertices and edges of an infinite path of T_0 starting with a_0.

4. Let T be an infinite tree, a ∈ {T}. If two infinite components are obtained by omitting an edge of T, then T cannot be 2-a-ordered.

Denote by (u, v) the omitted edge, and by T_u and T_v the two corresponding infinite components. Let there exist 2-a-ordering of T, and, without loss of generality, let u precede v. There is only a finite number of vertices in front of v; thus in the ordering.
there must necessarily occur, following \( v \), a vertex from \( T_u \) and a vertex from \( T_v \), distinct from \( u \) and \( v \). Hence one can find two neighbouring vertices \( x, y \) in the 2-a-ordering following \( v \) such that \( x \in \{ T_u \} \) and \( y \in \{ T_v \} \). As \( x \neq u \) and \( y \neq v \), there is \( \mu(x, y) \geq 3 \), which is a contradiction.

5. If it is possible to 2-a-order a finite or an infinite tree \( T \), then it is also possible to 2-a-order each of its subtrees \( U \) containing \( a \). Moreover, if we omit, in an arbitrary 2-a-ordering of \( T \), the vertices not belonging to \( U \), we obtain a 2-a-ordering of the subtree \( U \).

The statement can be proved as in [4]. Let \( \alpha \) be some 2-a-ordering of the tree \( T \). From the sequence \( \alpha \) form the subsequence \( \beta \) by omitting all vertices of \( \alpha \) not belonging to the subtree \( U \). In \( \beta \) there occur all vertices of \( U \) precisely once. Thus \( \beta \) is a 2-a-ordering of \( U \) if we show that the distance of every two neighbouring vertices in \( \beta \) is at most 2, in the metric of \( U \) (or \( T \)). Thus, let \( x \) and \( y \) be consequent vertices in \( \beta \). There are two cases possible:

a. The vertices \( x \) and \( y \) are consequent in \( \alpha \); then \( \mu(x, y) \leq 2 \) in \( T \) and also in \( U \).

b. Vertices \( x \) and \( y \) are not neighbouring in \( \alpha \); thus the 2-a-ordering \( \alpha \) of \( T \) has the form \( a_1, \ldots, x, z_1, z_2, \ldots, z_m, y, \ldots \) \( (m \geq 1) \), where \( z_i \in \{ U \} \) for \( i = 1, 2, \ldots, m \). Suppose that \( \mu(x, y) \geq 3 \) in \( U \) and hence also in \( T \). Let \( (x, v_1, v_2, \ldots, v_n, y) \) be the path of \( T \). Evidently \( n \geq 2 \) and all vertices \( v_j, j = 1, 2, \ldots, n \), belong to \( U \). By omitting the edge \( (v_1, v_2) \) from \( T \) there result two components \( X \) and \( Y \), \( x \in \{ X \}, y \in \{ Y \} \). Hence, between the vertices \( x, z_1, z_2, \ldots, z_m, y \) there exists at least one pair \( b, c \) of consequent vertices in \( \alpha \) such that \( b \in \{ X \} \) and \( c \in \{ Y \} \). As \( v_1 \) and \( v_2 \) do not coincide with any of the vertices \( x, z_1, z_2, \ldots, z_m, y \), there is also \( b \neq v_1 \) and \( c \neq v_2 \). The path \( (b, \ldots, c) \) necessarily contains \( v_1 \) and \( v_2 \) as inner vertices, therefore \( \mu(b, c) \geq 3 \). But this is a contradiction, because \( b, c \) are consequent in the 2-a-ordering \( \alpha \), i.e. \( \mu(b, c) \leq 2 \). Thus \( \mu(x, y) \geq 3 \) cannot occur, and therefore \( \mu(x, y) \leq 2 \).

6. Let \( T \) be an infinite tree, \( a \in \{ T \} \). If \( T \) contains more than one vertex of infinite degree, then it cannot be 2-a-ordered.

Let \( T \) be an infinite tree, \( a \in \{ T \} \). Let \( b \) be a vertex of denumerable degree of \( T \). Denote by \( Z \) the set of all those vertices of \( T \) of degree 1 (with the exception of \( a \)) which are connected by an edge to the vertex \( b \). Then \( T \) can be 2-a-ordered if an only if \( Z \) is non-empty and the subtree \( U \) obtained from \( T \) by omitting vertices of the set \( Z - \{ z_1 \} \) and all incident edges (where \( z_1 \) is an arbitrary vertex of \( Z \)) is a finite tree which can be 2-(a, \( z_1 \))-ordered.

Let \( T \) be an infinite tree not containing a vertex of an infinite degree, \( a \in \{ T \} \). Then, \( T \) can be 2-a-ordered if and only if for the subtree \( V \) obtained from \( T \) by omitting the vertices of degree 1 (except for the vertex \( a \)) and all incident edges, there holds:

0° there exists precisely one infinite path \( W \) of \( T \) starting with the vertex \( a \),

172
1° the degree of all vertices of $V$ is at most 4,
2° all vertices of degree 3 and 4 in $V$ are inner vertices of the path $W$,
3° between every two vertices of degree 4 in $V$ there lies at least one vertex of degree
2 in $T$; if a has degree greater than 1 in $T$, then between it and the nearest vertex
of degree 4 in $V$ there lies at least one vertex of degree 2 in $T$.

Necessity. Let $T$ be 2-a-orderable. Let $T$ contain vertices of an infinite degree.
According to item 4, $T$ can contain at most one vertex of denumerable degree; denote
it by $b$. Any infinite path cannot issue out of the vertex $b$ (e.g. $(b, b_1, b_2, \ldots)$) because,
again according to item 4, by omitting the edge $(b, b_1)$ from $T$ there would result two
infinite components, and $T$ could not be 2-a-orderable. From the vertex $b$ there can
issue only a finite number of disjoint paths of length greater than 1, since otherwise
there would exist a subtree of $T$ not 2-orderable (item 2 corollary, 1°), and thus,
according to item 5, $T$ itself could not be 2-a-orderable. As the vertex $b$ is of denu-
merable degree, the set $Z$ must be denumerable. Denote by $z_1, z_2, z_3, \ldots$ all vertices
of $Z$. Hence, if we omit from $T$ the vertices $z_2, z_3, \ldots$ and edges $(z_2, b), (z_3, b), \ldots$,
we obtain a finite subtree $U$. Consider some 2-a-ordering of $T$. As $\{U\}$ is a finite set,
some element of $\{U\} \setminus \{z_1\}$, say $u$, is the final member of this 2-a-ordering. Thus
the 2-a-ordering of $T$ is of the form $a, z_1, z_2, \ldots, z_{i_1}, \ldots, u, z_{i_2}, z_{i_3}, \ldots, z_{i_p}, z_{i_{p+1}}, z_{i_{p+2}}, \ldots$, where $i_1, i_2, \ldots$ is a suitable permutation of integers. Let $i_p = 1$. We assert
that then $a, u, z_1, z_2, \ldots, z_{i_p-1}, z_{i_p}, z_{i_{p+1}}, z_{i_{p+2}}, \ldots$ is also a 2-a-ordering of $T$.
This follows from the following consideration:

From the notation of the $z_j$ there follows $\mu(z_m, z_n) = 2$ for $m \neq n$, $m, n \geq 1$. Furthermore, either $u = b$, or $(u, b, z_j)$ is a path for all $j \geq 1$. Therefore $\mu(u, z_j) \leq 2$
for all $j \geq 1$. Thus, finally, if the 2-a-ordering of $T$ is of the form $a, \ldots, x, z_j, y, \ldots$,
then $\mu(x, y) \leq 2$ because $\mu(x, b) \leq 1$ and $\mu(b, y) \leq 1$.

Hence, the initial segment of the new 2-a-ordering of $T$, namely, $a, \ldots, u, z_1$, is a 2-(a, $z_1$)-ordering of its subtree $U$; therefore $U$ can be 2-(a, $z_1$)-ordered.

Now assume $T$ does not contain a vertex of infinite degree. Then, according to
item 3, an infinite path $W$ must issue from the vertex $a$ of $T$. This path is unique,
according to item 4. Indeed, consider another such path. As both belong $T$, the tree $T$
decomposes into components after omitting the vertex $a$, and at least two of these
(containing the considered infinite paths) would be infinite. Hence, according to
item 4, $T$ could not be 2-a-ordered. If, furthermore, condition 1° or 3° were not
satisfied, there would exist a finite subtree of the tree $T$ such that it would not satisfy
the corresponding condition 1° or 3° in item 2. Thus, this subtree would not be
2-a-orderable, and thus according to item 5, even $T$ would not be 2-a-orderable,
a contradiction. It remains to show that condition 2° is necessary, which we again
perform by means of a contradiction. Suppose that there exists some vertex $x$ of degree
3 or 4 in $V$ which is not an inner vertex of $W$. Choose a vertex $c \in \{W\}$ such
that the path $(a, \ldots, x)$ does not contain $c$. Let $T_0$ be the maximal subtree of $T$ con-
taining the vertices $a$ and $c$, in which the degree of $c$ is 1. Furthermore, let $\tau$ be
a 2-\(a\)-ordering of the tree \(T\). According to item 5, if we omit from \(\tau\) the vertices not belonging to \(T_0\), we obtain a 2-\(a\)-ordering of \(T_0\); denote this by \(\tau_0\). As \(\{W\}\) is infinite, there exists an infinite sequence of distinct vertices \(r_i \in \{W\}, i = 1, 2, \ldots\), following the vertex \(c\). For \(i = 1, 2, \ldots\) consider the maximal subtree \(T_i\) of \(T\) containing the vertices \(a\) and \(r_i\), the degree of the vertex \(r_i\) being equal to 1 in \(T_i\). By omitting the vertices not belonging to \(T_i\), we obtain from \(\tau\) a sequence \(\tau_i\) which is a 2-\(a\)-ordering of the tree \(T_i\). For \(i < j\), \(\tau_i\) is a proper subsequence of the sequence \(\tau_j\). Let the last vertex of the sequence \(\tau_i\) be \(t_i, i = 0, 1, 2, \ldots\). As \(\{T_0\}\) is finite, there exists an \(m\) such that \(t_m \in \{T_0\}\), i.e. the path \((a, \ldots, t_m)\) contains the vertex \(c\). In another words, at least one vertex of degree 3 or 4 in \(V\) is not an inner vertex of \((a, \ldots, t_m)\). On the other hand, \(T_m\) can be 2-\((a, t_m)\)-ordered. But this is a contradiction to the theorem of item 2, condition 2°.

**Sufficiency.** Let there be given a finite tree \(U\) which can be 2-\((a, z_1)\)-ordered, the degree of its vertex \(z_1\) is 1 and \((z_1, b)\) is its edge. Construct, from this tree \(U\), an infinite tree \(T\) by adding denumerable many vertices \(z_2, z_3, \ldots \in \{U\}\) and edges \((z_2, b)\), \((z_3, b)\), \ldots\). Let \(a, \ldots, z_1\) be some 2-\((a, z_1)\)-ordering of \(U\). Then \(a, \ldots, z_1, z_2, z_3, \ldots\) is a 2-\(a\)-ordering of \(T\) because \(\mu(z_i, z_{i+1}) = 2\) for \(i = 1, 2, 3, \ldots\).

Assume \(T\) does not contain a vertex of an infinite degree. By omitting all the vertices of degree 1 (except for \(a\)) and all incident edges from \(T\), obtain the tree \(V\) for which there holds 0°-3°.

Let the infinite path \(W\) of the tree \(T\) contain an infinite number of vertices of degree 2 in \(T\), say \(p_1, p_2, \ldots\). Let \(P_0\) denote the maximal subtree of \(T\) containing \(a\) and in which the vertex \(p_1\) has degree 1. Let \(P_i\) denote the maximal subtree of \(T\) in which the vertices \(p_i\) and \(p_{i+1}\) are of degree 1. According to item 2, there exists a 2-\((a, p_1)\)-ordering of the finite tree \(P_0\) (denote it by \(\pi_0\)), and for every \(i = 1, 2, \ldots\) there exists a 2-\((p_i, p_{i+1})\)-ordering of the finite tree \(P_i\) (denote it by \(\pi_i\)). Therefore, if \(\pi_i^*\) is the sequence obtained from \(\pi_i\) by omitting the last member \(p_{i+1}\), then \(\pi_0^*, \pi_1^*, \pi_2^*, \ldots\) is a 2-\(a\)-ordering of \(T\).

Let the path \(W\) contain only a finite number of vertices of degree 2 in \(T\); hence it contains only a finite number of vertices of degree 4 in \(V\). Thus, there exist denumerably many vertices of \(W\) with degree 2 or 3 in \(V\). Choose an infinite sequence \(v_1, v_2, \ldots\) from these, ordered by increasing distance from \(a\), and such that all vertices of degree 4 in \(V\) are between \(a\) and \(v_1\). Let \(T_0\) denote the maximal subtree of \(T\) such that it contains \(a\) and that \(v_1\) is of the degree 1 in it. Let \(T_i\) denote the maximal subtree of \(T\) such that, with the exception of vertices of the path \((v_i, \ldots, v_{i+1})\), it does not contain any vertex of the path \(W\), that it contains the vertices \(v_i\) and \(v_{i+1}\), and the degree of \(v_{i+1}\) is 1 in \(T_i\). According to the theorem of item 2, \(T_0\) can be 2-\((a, v_1)\)-ordered; denote by \(\tau_0^*\) one of these orderings in which we omit the last member, \(v_1\). According to the same theorem, \(T_i\) can be 2-\((v_i, v_{i+1})\)-ordered; denote by \(\tau_i^*\) one of these orderings, in which we omit the last member, \(v_{i+1}\) (\(i = 1, 2, \ldots\)). Evidently \(\tau_0^*, \tau_1^*, \tau_2^*, \ldots\) is a 2-\(a\)-ordering of the tree \(T\).
Výtah

USPOŘÁDÁNÍ UZLŮ NEKONEČNÉHO STROMU

František Neuman, Brno

Nekonečný strom \( T \) je neorientovaný souvislý graf bez kružnic se spočetně mnoha uzly. Množinu uzlů nekonečného stromu \( T \) označíme \( \{ T \} \). Nechť je \( a \in \{ T \} \). Říkáme, že takový strom lze 2-a-uspořádat, jestliže je možné seřadit uzly tohoto stromu v prostou posloupnost \( a = t_1, t_2, t_3, \ldots \) takovou, že vzdálenost mezi sousedními uzly v této posloupnosti je nejvýše dvě (v metrice stromu \( T \)).

V práci je dokázána nutná a postačující podmínka pro to, aby bylo možné daný nekonečný strom 2-a-uspořádat:

Nechť \( T \) je nekonečný strom, \( a \in \{ T \} \). Existuje-li v \( T \) více než jeden uzel nekonečného stupně, pak \( T \) nelze 2-a-uspořádat.

Nechť \( T \) je nekonečný strom, \( a \in \{ T \} \). Nechť \( b \) je uzel stromu \( T \), jehož stupeň je spočetný. Označme \( Z \) množinu všech uzlů stupně 1 stromu \( T \) s výjimkou uzlu \( a \), které jsou spojeny hranou s uzlem \( b \). Potom nekonečný strom \( T \) lze 2-a-uspořádat právě když je množina \( Z \) neprázdná a pro libovolnou posloupnost \( a = t_1, t_2, t_3, \ldots \) takovou, že vzdálenost mezi sousedními uzly posloupnosti je nejvýše dvě (v metrice stromu \( T \)).

V práci je dokázána nutná a postačující podmínka pro to, aby bylo možné daný nekonečný strom 2-a-uspořádat:

Nechť \( T \) je nekonečný strom, \( a \in \{ T \} \). Existuje-li v \( T \) více než jeden uzel nekonečného stupně, pak \( T \) nelze 2-a-uspořádat.

0° existuje právě jedna nekonečná cesta \( W \) stromu \( T \) začínající uzlem \( a \);
Резюме

УПОРЯДОЧЕНИЕ ВЕРШИН БЕСКОНЕЧНОГО ДЕРЕВА

ФРАНТИШЕК НЕЙМАН (František Neuman), Брно

Бесконечное дерево T — это неориентированный связный граф без циклов со счетным множеством вершин. Множество вершин бесконечного дерева T мы обозначим через \{T\}. Пусть a ∈ \{T\}. Мы говорим, что такое дерево можно 2-а-упорядочить, если можно упорядочить все вершины этого дерева в простую последовательность a = t_1, t_2, t_3, ... такую, что расстояние между соседними вершинами в этой последовательности не больше двух (в метрике дерева T).

В работе доказано необходимое и достаточное условие для того, чтобы данное бесконечное дерево было можно 2-а-упорядочить:

Пусть T — бесконечное дерево, a ∈ \{T\}. Если в T существует больше одной вершины бесконечной степени, то T нельзя 2-а-упорядочить.

Пусть T — бесконечное дерево, a ∈ \{T\}. Пусть b — вершина счетной степени дерева T. Мы обозначим через Z множество всех висячих вершин, за исключением вершины a, которые связаны ребром с вершиной b. Потом бесконечное дерево T можно 2-a-упорядочить тогда и только тогда, когда множество Z непусто и для произвольного z₁ ∈ Z то дерево U, которое мы получим удалением из дерева T вершин множества Z = \{z₁\} и с ними инцидентных ребер, есть конечное дерево, которое можно 2-(а, z₁)-упорядочить. (Пусть K — конечное дерево, a и b — его различные вершины. Конечное дерево K можно 2-(а, b)-упорядочить, если множество его вершин можно упорядочить в простую конечную последовательность a = t₁, t₂, ..., tₙ = b, такую, что расстояние между соседними вершинами в этой последовательности не больше двух (в метрике дерева K). Необходимое и достаточное условие 2-(a, b)-упорядочения см. в [2].)

Пусть T — бесконечное дерево, a ∈ \{T\}. Пусть в T не существует вершина бесконечной степени. Потом T можно 2-a-упорядочить тогда и только тогда, когда дерево V — которое мы получим из дерева T удалением всех висячих вершин дерева T, за исключением вершины a, и с ними инцидентных ребер — удовлетворяет следующим условиям:

1° ступень всех узлů stromu V je nejvýše 4;
2° všechny uzly stupně 3 a 4 ve V jsou vnitřními uzly cesty W;
3° mezi každými dvěma uzly stupně 4 ve V leží alespoň jeden uzel stupně 2 v T; pokud uzel a je stupně většího než 1 v T pak mezi ním a nejbližším uzlem stupně 4 ve V leží alespoň jeden uzel stupně 2 v T.
0° существует одна и только одна бесконечная цепь \( W \) дерева \( T \), начинающаяся в вершине \( a \),
1° степень всех вершин дерева \( V \) не больше 4,
2° все вершины степени 3 и 4 в \( V \) являются внутренними вершинами цепи \( W \),
3° между каждыми двумя вершинами степени 4 в \( V \) существует по крайней мере одна вершина степени 2 в \( T \); если степень вершины \( a \) больше 1 в \( T \), то между \( a \) и ближайшей вершиной степени 4 в \( V \) существует по крайней мере одна вершина степени 2 в \( T \).