Krzysztof Jarosz; Zbigniew Sawoń

A function algebra without any point derivation

Časopis pro pěstování matematiky, Vol. 111 (1986), No. 3, 230--234

Persistent URL: http://dml.cz/dmlcz/108162

Časopis pro pěstování matematiky, roč. 111 (1986), Praha

A FUNCTION ALGEBRA WITHOUT ANY POINT DERIVATION

KRZYSZTOF JAROSZ, ZBIGNIEW SAWOŃ, Warszawa

(Received May 18, 1983)

Let A be an algebra and let F be any homomorphism from A onto the scalar field. A functional Φ on A is called a derivation of A at the point F if

$$\Phi(fg) = F(f)\,\Phi(g) + \Phi(f)\,F(g)$$

for all f, g in A.

The notion of the point derivation has been studied for many years, especially in the case when A is a Banach algebra. For some classes of function algebras necessary and sufficient conditions for the existence of a point derivation are known (see e.g. [1], [2]). In 1967 J. Wermer gave an example of a compact subset K of the complex plane such that the function algebra $R(K)$ generated by rational functions with poles outside K is a proper subalgebra of $C(K)$ but there exists no continuous derivation at any point of K ([3]). In the same paper J. Wermer posed the following problem: Does there exist a Banach algebra B such that

(i) there is no continuous derivation at any point from the maximal ideal space of B,

(ii) the Shilov boundary of the algebra B is a proper subset of its maximal ideal space?

In this paper we give an example of a function algebra A (this is a commutative Banach algebra with unit in which every element f satisfies $\|f^2\| = \|f\|^2$) possessing both the above properties.

Theorem. *There exists a function algebra A such that*

(i) *there is no non-zero derivation at any point of the maximal ideal space of A,*

(ii) *the Shilov boundary of A is a proper subset of the maximal ideal space of A.*

Proof. We shall define A as a direct limit of a directed system $(A_\alpha, \varphi_{\alpha,\alpha'})_{\alpha\in\Gamma}$ of function algebras and algebra homomorphisms. The index set Γ is the product of the set of positive integers $\boldsymbol{N}$ and the set of all finite subsets of the unit disc $D = \{z \in \boldsymbol{C}: |z| < 1\}$.

If

$$\alpha = \{n_1, \{\lambda_1, \ldots, \lambda_k\}\} \quad \text{and}$$
$$\alpha' = \{n'_1, \{\lambda'_1, \ldots, \lambda'_l\}\}$$

then we define

$$\alpha \prec \alpha' \Leftrightarrow n_1 \mid n_1' \quad \text{and} \quad \{\lambda_1, \ldots, \lambda_k\} \subseteq \{\lambda_1', \ldots, \lambda_l'\}.$$

As the first step to obtain A_α we define algebras $\tilde{A}_\alpha$. If $\alpha = \{n, \{\lambda_1, \ldots, \lambda_k\}\}$ then $\tilde{A}_\alpha$ is a commutative algebra formally generated by the set of k generators $\{f_{n,\lambda_1}, \ldots \ldots, f_{n,\lambda_k}\}$ and the unit; this means $\tilde{A}_\alpha$ is the algebra of all finite formal series of the form

$$\sum a_\nu (f_{n,\lambda_1})^{\nu_1} \cdot \ldots \cdot (f_{n,\lambda_k})^{\nu_k} \tag{1}$$

where

$$\nu = (\nu_1, \ldots, \nu_k) \in (\{0\} \cup N)^k.$$

If $\alpha \prec \alpha'$ where

$$\alpha = \{n, \{\lambda_1, \ldots, \lambda_k\}\} \quad \text{and}$$

$$\alpha' = \{l \,.\, n, \{\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_.\}\},$$

then we define a map $\tilde{\varphi}^i_{\alpha,\alpha'}\colon \tilde{A}_\alpha \to \tilde{A}_{\alpha'}$:

$$\tilde{\varphi}^i_{\alpha,\alpha'}\left(\sum a_\nu (f_{n,\lambda_1})^{\nu_1} \cdot \ldots \cdot (f_{n,\lambda_k})^{\nu_k}\right) = \sum a_\nu (f_{l.n,\lambda_1})^{l.\nu_1} \cdot \ldots \cdot (f_{l.n,\lambda_k})^{l.\nu_k}.$$

It is easy to check that $\tilde{\varphi}^i_{\alpha,\alpha'}$ is an algebra homomorphism.

For each $\alpha \in \Gamma$ we now define a seminorm p_α on $\tilde{A}_\alpha$. For this purpose we denote by $r_n^k(z)$ the k-th branch of the n-th root of a complex number z $(0 \leqq k < n)$: $r_n^k(z) = \exp\left(1/n(\operatorname{Log} z + 2k\pi i)\right)$ where $\operatorname{Log} z$ is the main branch of the logarithm of z $(-\pi < \operatorname{Im} \operatorname{Log} z \leqq \pi)$.

If $f \in \tilde{A}_\alpha$ is of the form (1) then we define

$$p_\alpha(f) = \sup_{\substack{0 \leqq l_1 < n \\ \vdots \\ 0 \leqq l_k < n}} \sup_{z \in D} \left|\sum a_\nu (r_n^{l_1}(z - \lambda_1))^{\nu_1} \cdot \ldots \cdot (r_n^{l_k}(z - \lambda_k))^{\nu_k}\right|.$$

A simple computation shows that p_α is a seminorm on $\tilde{A}_\alpha$ and that for any $f \in \tilde{A}_\alpha$ we have

$$(p_\alpha(f))^2 = p_\alpha(f^2).$$

This proves that the completion of the algebra $\tilde{A}/\ker p_\alpha$ is a function algebra. We will denote it by A_α and its norm by $\| \; \|_\alpha$.

By the definition of $\tilde{\varphi}_{\alpha,\alpha'}$ we have

$$p_\alpha(f) = p_{\alpha'}(\tilde{\varphi}_{\alpha,\alpha'}(f)). \tag{2}$$

Thus $\tilde{\varphi}_{\alpha,\alpha'}$ defines an isometric embedding $\varphi_{\alpha,\alpha'}$ of A_α into $A_{\alpha'}$ which is also an algebra homomorphism. We set

$$(A, \|\cdot\|) = \overrightarrow{\lim_{\Gamma}}\left((A_\alpha, \|\cdot\|_\alpha), \varphi_{\alpha,\alpha'}\right).$$

To prove the first part of the theorem assume to the contrary that there exist a functional F from the maximal ideal space of A and a non-zero derivation $\Phi: A \to C$ at the point F.

The functional Φ being non-zero it is not equal to zero on some of the generators f_{n,λ_0} of some of the algebras A_{α_0}.

Notice that $(f_{2n,\lambda_0})^2 = f_{n,\lambda_0}$ in A, so the equality

$$\Phi(f_{n,\lambda_0}) = \Phi((f_{2n,\lambda_0})^2) = 2F(f_{2n,\lambda_0})\,\Phi(f_{2n,\lambda_0})$$

gives

$$F(f_{2n,\lambda_0}) \neq 0 \quad \text{and} \quad \Phi(f_{2n,\lambda_0}) \neq 0\,. \tag{3}$$

Notice also that $f_{1,0} - \lambda_0 1 = f_{1,\lambda_0}$ in A.

Hence

$$\begin{aligned} \Phi(f_{1,0}) &= \Phi(f_{1,0} - \lambda_0 \,.\, 1) = \Phi(f_{1,\lambda_0}) = \\ &= \Phi((f_{2n,\lambda_0})^{2n}) = 2n(F(f_{2n,\lambda_0}))^{2n-1}\,\Phi(f_{2n,\lambda_0}) \neq 0\,. \end{aligned} \tag{4}$$

Let $F(f_{1,0}) = c$. The norm of the element $f_{1,0}$ of the algebra A is equal to one, hence $|c| \leqq 1$.

Assume first that $|c| < 1$, then

$$(F(f_{2,c}))^2 = F((f_{2,c})^2) = F(f_{1,c}) = F(f_{1,0} - c\,.\,1) = 0$$

and we get

$$\Phi(f_{1,0}) = \Phi(f_{1,0} - c\,.\,1) = \Phi(f_{1,c}) = \Phi((f_{2,c})^2) = 2\,F(f_{2,c})\,\Phi(f_{2,c}) = 0\,,$$

which contradicts (4).

Suppose now that $|c| = 1$, and consider Φ and F as functionals on the subalgebra $A_{(1,0)}$ of A.

The algebra $A_{(1,0)}$ being the disc algebra $A(D)$ we arrive at the following conclusion: the functional Φ is a non-zero derivation of the disc algebra $A(D)$ at the point c from the boundary of the disc.

This is impossible ([1]) and the contradiction proves the first part of Theorem.

Now observe that by the definition of the norm $\| \; \|_\alpha$, for any complex number c of modulus not greater than one, there are linear and multiplicative functionals on A, assuming the value c for the element $f_{1,0}$.

It follows that in order to prove the second part of Theorem it is sufficient to show that if a functional F is in the Shilov boundary of A then $|F(f_{1,0})| = 1$.

Assume that this is not the case. Then it is not the case for some finitely generated subalgebra A_α of A, either. This means that there exists a linear and multiplicative functional F_0 on the algebra A_α such that $|F_0(f_{1,0})| < 1$ and such that F_0 is in the Shilov boundary of A_α.

Moreover, the density of the Choquet boundary in the Shilov boundary allows us to assume that F_0 is even in the Choquet boundary of A_α. Taking another α, if necessary, we can assume without loss of generality that $F_0(f_{1,0}) = \lambda_1$ and the index α of our subalgebra A_α is of the form $\alpha = \{n, \{\lambda_1, \lambda_2, \ldots, \lambda_k\}\}$.

From the definition of $\| \|_\alpha$ we see that the elemenets $(f_{n,\lambda_j})^n + \lambda_j . \mathbf{1}$ of the algebra A_α coincide for $j = 1, \ldots, k$. Thus there exist non-negative integers $l_1, l_2, \ldots, l_k$, all less than n, such that

$$F_0(f_{n,\lambda_j}) = r_n^{l_j}(\lambda_1 - \lambda_j) \quad \text{for} \quad j = 1, \ldots, k .$$

Let $\delta = \inf_{2 \leq j \leq k} |\lambda_1 - \lambda_j|$ and let

$$R_j: \left\{ z \in \mathbf{C}: |z + \lambda_1 - \lambda_j| \leqq \frac{\delta}{2} \right\} \to \mathbf{C}$$

be any analytic branches of the n-th root such that

$$R_j(\lambda_1 - \lambda_j) = r_n^{l_j}(\lambda_1 - \lambda_j) , \quad j = 2, 3, \ldots, k .$$

We define a homomorphism $\tilde{T}$ from the algebra $\tilde{A}_\alpha$ into the algebra of all analytic functions on the disc $D(0, \delta/2) = \{z \in \mathbf{C}: |z| \leqq \delta/2\}$:

$$\tilde{T}(\sum a_\nu (f_{n,\lambda_1})^{\nu_1} \cdot \ldots \cdot (f_{n,\lambda_k})^{\nu_k}) =$$
$$= \sum a_\nu z^{\nu_1} (R_2(z^n + \lambda_1 - \lambda_2))^{\nu_2} \cdot \ldots \cdot (R_k(z^n + \lambda_1 - \lambda_k))^{\nu_k} .$$

By the definition of p_α, for any f in $\tilde{A}_\alpha$ we have

$$\sup_{z \in D(0,\delta/2)} |\tilde{T}(f)(z)| \leqq p_\alpha(f) ,$$

hence $\tilde{T}$ gives a norm one homomorphism T from the algebra A_α into the disc algebra $A(D(0, \delta/2))$.

Composing T on the left with a usual function derivation at the point zero, we get a non-zero norm one derivation Φ on the algebra A_α at the point F_0.

To complete the proof it is sufficient to use the well-known fact that there never exists a continuous non-zero derivation at a point from the Choquet boundary of an algebra. Since the proof of this fact is very short and simple we recall it below instead of giving the references.

If F_0 is in the Choquet boundary of an algebra A then there is a net (g_i) contained in A such that

$$g_i(F_0) = 1 = \|g_i\| \quad \text{for all} \quad i \in I$$

and (g_i) tends uniformly to zero off any neighbourhood of F_0. Hence the net $(g_i . f_{1,\lambda_1})$ tends to zero in the norm. Consequently, we have

$$\lim \Phi(g_i . f_{1,\lambda_1}) = 0 .$$

However, on the other hand,

$$\Phi(g_i . f_{1,\lambda_1}) = F_0(g_i)\, \Phi(f_{1,\lambda_1}) + \Phi(g_i)\, F_0(f_{1,\lambda_1}) = \Phi(f_{1,\lambda_1}) = 1 .$$

References

[1] *A. Browder:* Point Derivations on Function Algebras, J. Func. Anal. *1* (1) 1967), 22—27.
[2] *A. P. Hallstrom:* On Bounded Point Derivations and Analytic Capacity, J. Func. Anal. *4* (1) 1969, 153—165.
[3] *J. Wermer:* Bounded Point Derivations on Certain Banach Algebras, J. Func. Anal. *1* (1) 1967, 28—36.

Authors' address: Institute of Mathematics, Warsaw University, Warsaw, Poland.