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REGULAR FACTORS IN POWERS OF CONNECTED GRAPHS

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Let $G$ be a graph (in the sense of [1] or [3]). We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. The number $|V(G)|$ is called the order of $G$. If $W \subseteq V(G)$, then we denote by $\langle W \rangle_G$ the subgraph of $G$ induced by $W$. If a spanning subgraph $F$ of $G$ is a regular graph of a degree $m \geq 0$, then we say that $F$ is an $m$-factor of $G$. For every integer $n \geq 1$, by the $n$-th power $G^n$ of $G$ we mean the graph with $V(G^n) = V(G)$ and

$$E(G^n) = \{ uv; u, v \in V(G) \text{ with the property that } 1 \leq d_G(u, v) \leq n \},$$

where $d_G$ denotes the distance between vertices in $G$.

If $n \geq 1$ is an odd integer and $G$ has an $n$-factor, then the order of $G$ is even. Chartrand, Polimeni and Stewart [2] and Sumner [6] proved that if $G$ is a connected graph of an even order, then $G^2$ has a 1-factor. Nebeský [4] proved that if $G$ is a connected graph of an even order $\geq 4$, then $G^4$ has a 3-factor. In the present paper these results will be generalized for every odd integer $n \geq 1$. We shall prove the following theorem:

**Theorem 1.** Let $n \geq 1$ be an odd integer, and let $G$ be a connected graph of an even order $p \geq n + 1$. Then $G^{n+1}$ has an $n$-factor.

In the present paper we shall prove one more theorem, which complements Theorem 1.

**Theorem 2.** Let $n \geq 2$ be an even integer, and let $G$ be a connected graph of an order $p \geq n + 1$. Then $G^{n+1}$ has an $n$-factor.

Let $G$ be the tree (homeomorphic with the star $K(1, n+2)$) of an order $p > n(n+1)$ which is given in Fig. 1. Then $G^n$ has no $n$-factor. This means that the value $n + 1$ of the power in Theorems 1 and 2 is the best possible.

Note that for $n = 2$ a stronger result is known. Sekanina [5] proved that if $G$ is a connected graph, then $G^3$ is hamiltonian connected.

To prove Theorem 1 and 2 we use two lemmas and three remarks.
Let $T$ be a nontrivial tree, and let $u$ and $v$ be adjacent vertices of $T$. Then $T - uv$ is a forest with exactly two components. We denote by $T(u, v)$ or $T(v, u)$ the component of $T - uv$ which contains $u$ or $v$, respectively.

![Diagram](image)

Fig. 1.

Let $T$ be a tree, and let $u \in V(T)$. We shall say that $W \subseteq V(T)$ is a $u$-set in $T$, if either $W = \{u\}$ or there exist distinct components $T_1, \ldots, T_i$ ($i \geq 1$) of $T - u$ such that either $W = V(T_1) \cup \ldots \cup V(T_i)$ or $W = \{u\} \cup V(T_1) \cup \ldots \cup V(T_i)$.

**Lemma 1.** Let $T$ be a tree of an order $p > n + 1$, where $n \geq 1$. Then there exists $u \in V(T)$ and disjoint $u$-sets $W'$ and $W''$ in $T$ such that

1. $W' \cup W'' \neq V(T)$ and $T - (W' \cup W'')$ is a tree;
2. $|W'| \leq n$ and $|W''| \leq n$;
3. $n < |W' \cup W''|$
4. if $|W' \cup W''| \neq n + 1$, then $|W' \cup W''|$ is even.

**Proof.** Since $p > n + 1$, there exist adjacent vertices $u$ and $v$ such that $|V(T(u, v))| > n$ and $|V(T(w, u))| \leq n$ for every vertex $w \neq v$ such that $uw \in E(T)$.

(Otherwise, in $T$ we can construct an infinite sequence of distinct vertices beginning in an arbitrary vertex of degree one, which contradicts the finiteness of $V(T)$).

Let $T_1, \ldots, T_i$ ($i \geq 1$) be all the components of $T - u$ which are different from $T(v, u)$. Denote $M_1 = V(T_1), \ldots, M_i = V(T_i)$ and $m = |M_1| + \ldots + |M_i|$. Clearly, $m = |V(T(u, v))| - 1$. Without loss of generality we assume that

$$n \geq |M_1| \geq \ldots \geq |M_i| > 0.$$ 

Since $|V(T(u, v))| > n$, we have $m \geq n$. We shall construct disjoint $u$-sets $W'$ and $W''$ with the properties (1)–(4). We distinguish the following cases and subcases:

1. $m = n$. We put $W' = M_1 \cup \ldots \cup M_i$ and $W'' = \{u\}$.
2. $m > n$. It is obvious that there exists an integer $f$, $1 \leq f < i$, such that
\( (n + 1)/2 \leq |M_1| + \ldots + |M_f| \leq n. \)

Denote \( m_1 = |M_1| + \ldots + |M_f|. \)

2.1. \( m - m_1 \leq n. \) If \( m \) is even, then we put \( W' = M_1 \cup \ldots \cup M_f \) and \( W'' = M_{f+1} \cup \ldots \cup M_i. \) Assume that \( m \) is odd. If \( m_1 < m - m_1 \), then we put \( W' = \{u\} \cup M_1 \cup \ldots \cup M_f \) and \( W'' = M_{f+1} \cup \ldots \cup M_i. \) If \( m - m_1 < m_1 \), then we put \( W' = M_1 \cup \ldots \cup M_f \) and \( W'' = \{u\} \cup M_{f+1} \cup \ldots \cup M_i. \)

2.2. \( m - m_1 > n. \) Then there exists \( g, f < g < i, \) such that

\( (n + 1)/2 \leq |M_{f+1}| + \ldots + |M_g| \leq n. \)

Denote \( m_2 = |M_{f+1}| + \ldots + |M_g|. \)

2.2.1. \( m_1 + m_2 \) is even. Then we put \( W' = M_1 \cup \ldots \cup M_f \) and \( W'' = M_{f+1} \cup \ldots \cup M_i. \)

2.2.2. \( m_1 + m_2 \) is odd.

2.2.2.1. \( m - m_1 \) is even. Then we put \( W' = M_{f+1} \cup \ldots \cup M_g \) and \( W'' = M_{g+1} \cup \ldots \cup M_i. \)

2.2.2.2. \( m - m_1 \) is odd. Then \( m - m_2 \) is even.

2.2.2.1.1. \( m - m_1 > n. \) Then \( m_1 + (m_1 + m_2) > n. \) We put \( W' = M_1 \cup \ldots \cup M_f \) and \( W'' = M_{g+1} \cup \ldots \cup M_i. \)

2.2.2.2.2. \( m - m_1 \leq n. \) If \( m \) is even, then we put \( W' = M_1 \cup \ldots \cup M_f \cup \ldots \cup M_g \cup \ldots \cup M_i \) and \( W'' = M_{f+1} \cup \ldots \cup M_g. \) Assume that \( m \) is odd. If \( m_2 < m - m_2, \) then we put \( W' = M_1 \cup \ldots \cup M_f \cup M_{g+1} \cup \ldots \cup M_i \) and \( W'' = \{u\} \cup M_{f+1} \cup \ldots \cup M_g. \) If \( m_2 > m - m_2, \) then we put \( W' = M_{f+1} \cup \ldots \cup M_f \cup M_{g+1} \cup \ldots \cup M_i \) and \( W'' = M_{f+1} \cup \ldots \cup M_g. \)

2.2.2.2.2. \( m - (m_1 + m_2) > n. \) Then there exists an integer \( h, g < h < i, \) such that

\( (n + 1)/2 \leq |M_{g+1}| + \ldots + |M_h| \leq n. \)

Denote \( m_3 = |M_{g+1}| + \ldots + |M_h|. \)

2.2.2.2.1. \( m_3 + m_1 \) is even. Then we put \( W' = M_1 \cup \ldots \cup M_f \) and \( W'' = M_{g+1} \cup \ldots \cup M_h. \)

2.2.2.2.2. \( m_3 + m_1 \) is odd. Then \( m_3 + m_2 \) is even. We put \( W' = M_{f+1} \cup \ldots \cup M_g \) and \( W'' = M_{g+1} \cup \ldots \cup M_h. \)

The proof of the lemma is complete.

Remark 1. Let \( T \) be a tree, \( u \in V(T), n \geq 1, \) and let \( W_1, \ldots, W_k \) \((k \geq 2)\) be disjoint \( u \)-sets such that \(|W_1| \leq n, \ldots, |W_k| \leq n. \) Then every set \( W_h, 1 \leq h \leq k, \) can be arranged into a sequence \( w_{h,1}, \ldots, w_{h,|W_h|} \) such that, for every \( g, 1 \leq g \leq |W_h|, \)

\[ d_T(w_{h,g}, u) < g, \quad \text{and} \quad d_T(w_{h,g}, u) \leq g. \]

This means that if \( u \in W_h, \) then \( w_{h,1} = u. \)

Let \( h' \) and \( h'' \) be arbitrary integers such that \( 1 \leq h' < h'' \leq k. \) Assume that \( g' \) and \( g'' \) are integers such that \( 1 \leq g' \leq |W_{h'}| \) and \( 1 \leq g'' \leq |W_{h''}| \) and that \( u \in W_{h''} \cup \ldots \cup W_{h'} \).

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Then $d_T(w^i, w^j) = d_T(w^i, w^j + 1) + d_T(w^j, w^i + 1)$.

Let $W$ be a finite nonempty set. Then we denote by $K(W)$ the complete graph whose vertex set is $W$.

Remark 2. Let $T$ be a tree, $n \geq 1$, and let $w_1, \ldots, w_m$ be a sequence of distinct vertices in $T$ which has been obtained in the way described in Remark 1. Let $m$ be even and $n + 1 \leq m \leq 2n$. Denote

$$E_0 = \{w_1w_{(m/2) + 1}, w_1w_{(m/2) + 2}, \ldots, w_1w_{n + 1}, w_2w_{(m/2) + 2}, w_2w_{(m/2) + 3}, \ldots, w_2w_{n + 2}, \ldots, w_{m/2}w_m, w_{m/2}w_{m + 1}, \ldots, w_{m/2}w_{n + (m/2)}\},$$

where every index $i > m$ is to be replaced by the index $i - (m/2)$. We denote by $F$ the graph with $V(F) = \{w_1, \ldots, w_m\}$ and

$$E(F) = E(K(\{w_1, \ldots, w_{m/2}\})) \cup E(K(\{w_{(m/2) + 1}, \ldots, w_m\})) \cup E_0.$$

Then $F$ is an $n$-factor of the graph $\langle\{w_1, \ldots, w_m\}\rangle_{T^{n+1}}$.

Remark 3. Let $m$ and $n$ be integers such that $0 < m < n$. It follows from Theorems 9.1 and 9.6 in [3] that $K_n$ has an $m$-factor if and only if at least one of the integers $m$ and $n$ is even.

Lemma 2. Let $T$ be a tree of an order $p \geq n + 1$, where $n \geq 1$. Assume that if $n$ is odd, then $p$ is even. Then $T^{n+1}$ has an $n$-factor.

Proof. If $p = n + 1$, then $T^{n+1} = K(V(T))$ and thus $T^{n+1}$ is a regular graph of the degree $n$. Assume that $p > n + 1$, and that for every tree $T^*$ of an order $p^*$, where (i) $n + 1 \leq p^* < p$, and (ii) if $n$ is odd, then $p^*$ is even, it is proved that $(T^*)^{n+1}$ has an $n$-factor. Since $p > n + 1$, it follows from Lemma 1 that there exists $u \in V(T)$ and disjoint $u$-sets $W'$ and $W''$ which fulfil (1)–(4). Clearly, if $n$ is odd, then $|V(T)|$ and $|W' \cup W''|$ are even, and therefore $|V(T) - (W' \cup W'')|$ is also even.
First, assume that \(|V(T) - (W' \cup W'')| \geq n + 1\). The induction assumption yields that \( (T - (W' \cup W'))_{T + 1} \) has an \( n \)-factor. If \( |W' \cup W''| = n + 1 \), then \( (W' \cup W')_{T + 1} \) and thus \( T_{T + 1} \) has an \( n \)-factor. Let \( |W' \cup W''| > n + 1 \). Then \( W' \cup W'' \) is even. The set \( W' \cup W'' \) can be arranged into a sequence \( w_1, \ldots, w_m \) described in Remark 1. From this fact and from Remark 2 it follows that there exists an \( n \)-factor of the graph \( (W' \cup W'')_{T + 1} \). Hence, \( T_{T + 1} \) has an \( n \)-factor.

We now assume that \( |V(T) - (W' \cup W'')| \leq n \). We distinguish the following cases and subcases:

1. There exist disjoint \( u \)-sets \( W_1 \) and \( W_2 \) such that \( |W_1| \leq |W_2| \leq n \) and that \( W_1 \cup W_2 = V(T) - \{u\} \).
   1.1. \( p \) is even. Then \( |W_1| < |W_2| \) and \( |W_1 \cup \{u\}| \leq n \). The set \( \{u\} \cup W_1 \cup W_2 \) can be arranged into a sequence \( w_1, \ldots, w_m \) (where \( m = p \)) described in Remark 1. Since \( n + 2 \leq m \leq 2n \) and \( m \) is even, it follows from Remark 2 that there exists an \( n \)-factor \( T_{T + 1} \).

1.2. \( p \) is odd. Then \( n \) is even. The set \( W_1 \cup W_2 \) can be arranged into a sequence \( w_1, \ldots, w_m \) (where \( m = p - 1 \)) described in Remark 1. Since \( n \) is even, we have that \( n + 2 \leq m \leq 2n \). Consider the graph \( F \) defined in Remark 2. Since \( m \geq n + 2 \), there exist positive even integers \( i \leq m/2 \) and \( j \leq m/2 \) such that \( i + j = n \).

Let \( F' \) be the graph obtained from the graph

\[
F = \{ w_1w_2, w_3w_4, \ldots, w_{i-1}w_i, \ldots, w_{m/2+1}w_{m/2+2}, \\
w_{(m/2)+3}w_{(m/2)+4}, \ldots, w_{(m/2)+j}w_{(m/2)+j+1}, \ldots, uw_{(m/2)+j} \}
\]

by adding the vertex \( u \) and the edges \( uw_1, uw_2, \ldots, uw_i, uw_{(m/2)+1}, uw_{(m/2)+2}, \ldots, uw_{(m/2)+j} \). Then \( F' \) is an \( n \)-factor of \( T_{T + 1} \).

2. For arbitrary disjoint \( u \)-sets \( W_1 \) and \( W_2 \) such that \( |W_1| \leq n \) and \( |W_2| \leq n \) it holds that \( W_1 \cup W_2 \neq V(T) - \{u\} \). Since \( |W'| \leq n \), \( |W''| \leq n \), and \( |V(T) - (W' \cup W'')| \leq n \), we conclude that there exist disjoint \( u \)-sets \( A \), \( B \), and \( C \) such that \( |A| \leq n \), \( |B| \leq n \), \( |C| \leq n \), \( |A \cup B| > n \), \( |B \cup C| > n \), \( |A \cup C| > n \), and \( A \cup B \cup C = V(T) - \{u\} \). Denote \( a = |A| \), \( b = |B| \), and \( c = |C| \). Without loss of generality we assume that \( a \geq b \geq c \).

2.1. Either \( a + b \) is odd or \( c < b \). If \( a + b \) is odd, then \( n \geq a > b \), and we put \( \bar{A} = A \), \( \bar{B} = B \cup \{u\} \), and \( \bar{C} = C \); if \( a + b \) is even, then \( c < b \), and we put \( \bar{A} = A \), \( \bar{B} = B \) and \( \bar{C} = C \cup \{u\} \). Denote \( \bar{a} = |\bar{A}| \), \( \bar{b} = |\bar{B}| \), and \( \bar{c} = |\bar{C}| \). Thus \( n \geq \bar{a} \geq \bar{b} \geq \bar{c} \), \( \bar{b} + \bar{c} > n \), and \( \bar{a} + \bar{b} \) is even. In accordance with Remark 1 the set \( \bar{C} \) can be arranged into a sequence \( z_1, \ldots, z_{\bar{c}} \) such that for every \( g \), \( 1 \leq g \leq \bar{c} \), \( u \in \bar{C} \) implies \( d_T(z_g, u) < g \) and \( u \notin \bar{C} \) implies \( d_T(z_g, u) \leq g \) (hence, if \( u \in \bar{C} \), then \( z_1 = u \)). Analogously we can arrange the sets \( \bar{A} \) and \( \bar{B} \). Moreover, in accordance with Remark 1, the set \( \bar{A} \cup \bar{B} \) can be arranged into a sequence \( w_1, \ldots, w_m \) (where \( m = \bar{a} + \bar{b} \)) with the properties described in Remark 1 and such that \( w_1, \ldots, w_g \in \bar{A} \) and \( w_{g+1}, \ldots, w_m \in \bar{B} \) (if \( u \in \bar{B} \), then \( w_{g+1} = u \)). According to Remark 1, for \( 1 \leq i \leq \bar{c} \) and \( 1 \leq j \leq \bar{b} \), the inequality \( i + j \leq n + 2 \) implies \( d_T(z_i, w_{g+j}) \leq n + 1 \). Let \( F \) be the regular graph constructed in Remark 2. Thus \( V(F) = \{w_1, \ldots, w_m\} \).
Let \( \bar{c} \) be odd; since \( p = \bar{a} + \bar{b} + \bar{c} \) and \( \bar{a} + \bar{b} \) is even, we have that \( p \) is odd and therefore \( n \) is even. This means that at least one of the integers \( \bar{c} \) and \( n \) is even. Thus at least one of the integers \( \bar{c} \) and \( n - \bar{c} + 1 \) is even.

2.1.1. \( \bar{c} < (n + 1)/2 \). Since \( \bar{b} + \bar{c} \geq n + 1 \), we have \( m - \bar{a} = \bar{d} \geq n - \bar{c} + 1 > \bar{c} \). It follows from Remark 3 that \( K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+1+n-\bar{c}}\}) \) has a \( \bar{c} \)-factor, say \( H_1 \). This means that the graph obtained from the graphs \( F - E(H_1) \) and \( K(\bar{C}) \) by adding the edges

\[
\begin{align*}
Z_{\bar{c}}w_{\bar{a}+1}, & \quad Z_{\bar{c}}w_{\bar{a}+2}, \ldots, Z_{\bar{c}}w_{\bar{a}+1+n-\bar{c}}, \\
Z_{\bar{c}-1}w_{\bar{a}+1}, & \quad Z_{\bar{c}-1}w_{\bar{a}+2}, \ldots, Z_{\bar{c}-1}w_{\bar{a}+1+n-\bar{c}}, \\
& \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
Z_{\bar{a}+1}, & \quad Z_{\bar{a}+2}, \ldots, Z_{\bar{a}+1+n-\bar{c}}
\end{align*}
\]

is an \( n \)-factor of \( T^{n+1} \).

2.1.2. \( c > (n + 1)/2 \). Then \( n - \bar{c} + 1 < \bar{c} \leq \bar{b} \). According to Remark 3, \( K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+\bar{c}}\}) \) has an \((n - \bar{c} + 1)\)-factor, say \( H_2 \). The graph obtained from the graphs \( F - E(H_2) \) and \( K(\bar{C}) \) by adding the edges

\[
\begin{align*}
Z_{\bar{c}}w_{\bar{a}+1}, & \quad \cdots, Z_{\bar{c}}w_{\bar{a}+1+n-\bar{c}}, \\
Z_{\bar{c}-1}w_{\bar{a}+1}, & \quad \cdots, Z_{\bar{c}-1}w_{\bar{a}+2+n-\bar{c}}, \\
& \quad \cdots \cdots \cdots \cdots \cdots \cdots \\
Z_{\bar{a}+\bar{c}}, & \quad \cdots, Z_{\bar{a}+n}
\end{align*}
\]

where every index \( i > \bar{a} + \bar{c} \) is to be replaced by the index \( i - \bar{c} \), is an \( n \)-factor of \( T^{n+1} \).

2.1.3. \( \bar{c} = (n + 1)/2 \). Then \( n \) is odd, and thus \( \bar{c} \) is even. Obviously, \( \bar{c} = n - \bar{c} + 1 \). We denote by \( d \) the integer \( \bar{a} \) if \( u \notin \bar{B} \), or the integer \( \bar{a} + 1 \) if \( u \in \bar{B} \). Obviously, \( m - d \geq \bar{c} \). We denote by \( d' \) the integer \( d - 1 \) and \( d \) which has the same parity as \( m/2 \). It is not difficult to see that \( d' \leq \bar{c} \). For every \( i, 1 \leq i \leq \bar{c} \), we have \( d'(z_i, w_{d'-\bar{c}+1}) \leq d'(z_i, w_{d-1-\bar{c}+1}) \leq n + 1 \). The graph obtained from the graphs \( K(\bar{C}) \) and

\[
F - E(K(\{w_{\bar{a}+1}, \ldots, w_{\bar{a}+\bar{c}}\})) - \{w_{d', w_{d'-1}, w_{d'-2}, w_{d'-3}, \ldots, w_{d'-\bar{c}+1}}\}
\]

by adding the edges

\[
\begin{align*}
Z_{\bar{c}}w_{d+1}, & \quad \cdots, Z_{\bar{c}}w_{d+\bar{c}+1}, \\
Z_{\bar{a}+1}, & \quad \cdots, Z_{\bar{a}+d+2-\bar{c}}
\end{align*}
\]

where every index \( i > d + \bar{c} \) means \( i - \bar{c} \), and the edges

\[
Z_{\bar{c}}w_{d'}, Z_{\bar{a}-1}w_{d'-1}, \ldots, Z_{\bar{a}+\bar{c}+1}
\]

is an \( n \)-factor of \( T^{n+1} \).
2.2. \(a + b\) is even and \(c = b\). Thus \(c \geq (n + 1)/2\). Since \(p = a + 2c + 1\), \(p + c\) is odd. This means that if \(c\) is even, then \(n\) is even. The set \(A \cup B\) can be arranged into a sequence \(w_1, \ldots, w_m\) (where \(m = a + c\)) with the properties described in Remark 1 and such that \(w_1, \ldots, w_a \in A\) and \(w_{a+1}, \ldots, w_m \in B\). The set \(C\) can be arranged into a sequence \(z_1, \ldots, z_c\) such that \(d(T, u) \leq i\) for every \(1, 1 \leq i \leq c\). Let \(F\) be the graph defined in Remark 2. Hence \(V(F) = \{w_1, \ldots, w_m\}\).

2.2.1. \(n\) is even. Then \(c = (n + 1)/2\). This means that \(c > (n + 1)/2\) and therefore \(n - c + 1 < c\). This means that either \(c\) or \(n - c + 1\) is even. It follows from Remark 3 that \(K(\{w_{a+1}, \ldots, w_{a+c}\})\) has an \((n - c + 1)\)-factor, say \(H_1\). Let \(F_1\) be the graph obtained from the graphs \(K(C)\) and \(F - E(H_1)\) by adding the edges

\[
\begin{align*}
&z_cw_{a+1}, \ldots, z_cw_{a+n-c+1}, \\
&\ldots \\
&z_1w_{a+c}, \ldots, z_1w_{a+n},
\end{align*}
\]

where every index \(i > a + c\) means \(i - c\). It is easy to see that \(F_1\) is an \(n\)-factor of \(\langle V(T - u) \rangle_{T^{n+1}}\). Since \(m/2 \geq c > (n + 1)/2\), there exist positive even integers \(j \leq m/2\) and \(k \leq c\) such that \(j + k = n\). The graph obtained from the graph

\[
F_1 = \{w_1w_2, w_3w_4, \ldots, w_{j-1}w_j, z_1z_2, z_3z_4, \ldots, z_{k-1}z_k\}
\]

by adding the vertex \(u\) and the edges

\[
uw_1, uw_2, \ldots, uw_j, uz_1, uz_2, \ldots, uz_k
\]

is an \(n\)-factor of \(T^{n+1}\).

2.2.2. \(n\) is odd. Then \(c\) is odd and therefore \(n - c\) is even. Since \(c \geq (n + 1)/2\), we have \(n - c < c\). Since \(n - c\) is even, we have that \(K(\{w_{a+1}, \ldots, w_{a+c}\})\) has an \((n - c)\)-factor, say \(H_2\). Let \(F_2\) be the graph obtained from the graphs \(F - E(H_2)\) and \(K(C)\) by adding the edges

\[
\begin{align*}
&z_cw_{a+1}, \ldots, z_cw_{a+n-c}, \\
&\ldots \\
&z_1w_{a+c}, \ldots, z_1w_{a+n-1},
\end{align*}
\]

where every index \(i > a + c\) means \(i - c\). Therefore, every vertex \(w_j, 1 \leq j \leq m\), has the degree \(n\) in \(F_2\), and every vertex \(z_k, 1 \leq k \leq c\), has the degree \(n - 1\) in \(F_2\). Obviously, \(n - c < m/2\). The graph obtained from the graph

\[
F_2 = \{w_1w_2, w_3w_4, \ldots, w_{n-c-1}w_{n-c}\}
\]

by adding the edges

\[
uw_1, \ldots, uw_{n-c}, uz_1, \ldots, uz_c
\]

is an \(n\)-factor of \(T^{n+1}\).

Thus the lemma is proved.
Proof of Theorems 1 and 2. Let \( G \) be a graph satisfying the conditions of Theorems 1 or 2. Then \( G \) is connected, and thus there exists a spanning tree of \( G \), say \( T \). According to Lemma 2, \( T^{n+1} \) has an \( n \)-factor. Thus \( G^{n+1} \) has an \( n \)-factor, which completes the proof.

References


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