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GLOBAL EXISTENCE OF SOLUTIONS OF CERTAIN FUNCTIONAL-DIFFERENTIAL EQUATIONS

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In this paper we are concerned with the problem of the existence and uniqueness of solutions of the following functional-differential equations

(1) $$y'(x) = f(x, y(h_1(x)), \ldots, y(h_n(x)), y'(h_{n+1}(x)), \ldots, y'(h_{n+m}(x)), u),$$

in the interval $J = (-\alpha, \alpha)$, $0 < \alpha \leqq \infty$, where h_i, $i = 1, 2, \ldots, n + m$ are continuous on J into J, f is a continuous vector-valued function on $J \times E^k \times \ldots \times E^k \times E$ into E^k satisfying the global Lipschitz condition

(2) $$|f(x, y_1, \ldots, y_{n+m}, u) - f(x, y_1^*, \ldots, y_{n+m}^*, u)| \leqq \sum_{i=1}^{n+m} z_i(x) |y_i - y_i^*|$$

for every $(y_1, \ldots, y_{n+m})$, $(y_1^*, \ldots, y_{n+m}^*) \in (E^k)^{n+m}$, $x \in J$, $u \in E$, and nonnegative continuous functions $z_i(x)$, $i = 1, 2, \ldots, n + m$ defined on J so that:

(i) there exists a constants $K > 0$ such that $z_i(x) \leqq K$ for $i = 1, 2, \ldots, n$, for all $x \in J$;

(ii) there exists a constant C, $0 \leqq C < 1$ such that $\sum_{i=n+1}^{n+m} z_i(x) \leqq C$ for all $x \in J$.

Here $|\cdot|$ denotes the usual norm in E^k.

An equation of this has been studied under different assumptions by many authors; see, e.g. [6], [7]. The exact form of (1) has been investigated, under still different assumptions, by ST. CZERWIK in ([3], [4]) where he takes a general Banach space instead of E^k.

We use here the ideas of [2] employed in [5] to establish a theorem on the existence of a unique solution of (1). We shall also consider the problem of continuous dependence of solutions of (1) on a parameter u. The results of this paper extend that of [5].

Following [5], we let $\{I_j \mid j \geqq 1\}$ be an increasing family of compact intervals which contain zero and $\bigcup_j I_j = J$. We denote by $c(I_j)$ the Banach space of continuous vector-valued functions $g : I \to E^k$ with norm

(3) $$\|g\|_{(j,\lambda)} = \sup_{x \in I_j} \{\exp(-\lambda|x|)\, |g(x)|\},$$

where λ is an arbitrary parameter. The Fréchét space $c(J)$ may be topologized by the family of seminorms $\{\|g\|_{(j,\lambda)} \mid j \geqq 1\}$. If $\lambda = 0$, the spaces $c(I_j)$ have the usual sup norm $\|\cdot\|_0$ on I_j.

Theorem 1. *If the function $f(x, y_1, \ldots, y_{n+m}, u)$ satisfies (2), and if*

(4) $$x\, h_i(x) \geqq 0\,, \quad |h_i(x)| \leqq |x|\,, \quad x \in J\,, \quad i = 1, 2, \ldots, n + m\,,$$

then the initial value problem $y(0) = y_0$ has a unique solution y for every $y_0 \in E^k$, which is given as the limit of successive approximations.

Proof. Let I be a compact subinterval of J containing zero and for simplicity, denote the norm of $g \in c(I)$ by $\|g\|_\lambda$. From (3), it follows that the norms $\|g\|_\lambda$, for arbitrary real λ, are all equivalent to the norm $\|g\|_0$. The identity

(5) $$\left|\int_0^x \exp(\lambda|t|)\, dt\right| = \frac{1}{\lambda}\{\exp(\lambda|x|) - 1\}$$

is valid for every $x \in J$, $\lambda > 0$.

We shall reduce our problem by substitution $g(x) = y'(x)$, $(y(x) = y_0 + \int_0^x g(s)\, ds)$ to the following equation

(6) $$g(x) = f(x, y_0 + \int_0^{h_1(x)} g(s)\, ds, \ldots,$$
$$y_0 + \int_0^{h_n(x)} g(s)\, ds, g(h_{n+1}(x)), \ldots, g(h_{n+m}(x)), u)\,.$$

Let $u \in E$ be fixed. It is obvious the transformation $\Phi = T(g)$ defined by the right-hand side of (6) maps $c(I)$ continuously into itself. We shall prove that

(7) $$\|Tg_2 - Tg_1\|_\lambda \leqq \left(\frac{nK}{\lambda} + C\right) \|g_2 - g_1\|_\lambda$$

for all $g_1, g_2 \in c(I)$ and $\lambda > 0$. Using (2) and the definition of $\|\cdot\|_\lambda$ we have:

$$|Tg_2(x) - Tg_1(x)| \leqq \sum_{i=1}^{n} z_i(x) \left|\int_0^{h_i(x)} (g_2(s) - g_1(s))\, ds\right| +$$
$$+ \sum_{i=n+1}^{n+m} z_i(x)\, |g_2(h_i(x)) - g_1(h_i(x))| \leqq$$
$$\leqq \sum_{i=1}^{n} z_i(x) \left|\int_0^x |g_2(s) - g_1(s)|\, ds\right| + \sum_{i=n+1}^{n+m} z_i(x)\, \left|g_2(h_i(x)) - g_1(h_i(x))\right| \leqq$$

$$\leqq \sum_{i=1}^{n} z_i(x) \|g_2 - g_1\|_\lambda \left| \int_0^x \exp(\lambda|s|)\, ds \right| +$$

$$+ \sum_{i=n+1}^{n+m} z_i(x) \|g_2 - g_1\|_\lambda \exp(\lambda|h_i(x)|) \leqq$$

$$\leqq \|g_2 - g_1\|_\lambda \left(\frac{nK}{\lambda} + C\right) \exp(\lambda|x|),$$

where we have used (4) and ((4), (5)) to obtain the second and the fourth inequalities respectively. Thus

$$\|Tg_2 - Tg_1\|_\lambda \leqq \left(\frac{nK}{\lambda} + C\right) \|g_2 - g_1\|_\lambda .$$

Now choose $\lambda > 0$ so that $nK/\lambda + C < 1$ and apply the classical Banach contraction principle to T and the distance function $\|g_2 - g_1\|_\lambda$ to complete the proof.

Now we consider the problem of continuous dependence of solutions of our problem on a parameter u.

Theorem 2. *Let the hypotheses of Theorem* 1 *be satisfied. If there exist a constant* M *and a function* $G : J \to J$ *such that for every* $x \in J$, $u, u_1 \in E$, $(y_1, \ldots, y_{n+m}) \in \in (E^k)^{n+m}$

$$|f(x, y_1, \ldots, y_{n+m}, u) - f(x, y_1, \ldots, y_{n+m}, u_1)| \leqq G(x)\, |u - u_1|$$

and

$$\sup_{x \in J} \{\exp(-\lambda|x|)\, G(x)\} \leqq M ,$$

then solutions $y(x, u)$ *of* (1) *fulfilling* $y(0, u) = y_0$ *is continuous with respect to the variables* (x, u) *in* $J \times E$.

Proof. For $g \in c(I)$ we define the transformation $T_u(g)$ by the right-hand side of the equation (6). From (7) we have

$$\|T_u(g) - T_u(y)\|_\lambda \leqq \left(\frac{nK}{\lambda} + C\right) \|g - y\|_\lambda .$$

From the hypotheses we obtain

$$\exp(-\lambda|x|)\, |T_u(g)(x) - T_{u_1}(g)(x)| \leqq G(x)\, |u - u_1| \exp(-\lambda|x|)$$

and hence

$$\|T_u(g) - T_{u_1}(g)\|_\lambda \leqq M|u - u_1| .$$

From theorem 1, there exist unique function $g(x, u)$, $g(., u) \in c(J)$ such that

$$y(x, u) = y_0 + \int_0^x g(s, u)\, ds ,$$

$$T_u(g(x, u)) = g(x, u), \ T_{u_1}(g(x, u_1)) = g(x, u_1) \quad \text{for} \quad x \in J .$$

Therefore, we have

$$\|g(x, u) - g(x, u_1)\|_\lambda \leqq \|T_u(g(x, u)) - T_u(g(x, u_1))\|_\lambda +$$

$$+ \|T_u(g(x, u_1)) - T_{u_1}(g(x, u_1))\|_\lambda \leqq \left(\frac{nK}{\lambda} + C\right) \|g(x, u) - g(x, u_1)\|_\lambda + M|u - u_1| .$$

Hence

$$\|g(x, u) - g(x, u_1)\|_\lambda \leqq \left(1 - \left(\frac{nK}{\lambda} + C\right)\right)^{-1} M|u - u_1| .$$

Consequently the function g is continuous with respect to the variable $x \in J$, uniformly with respect to the variable $u \in E$; so y is also continuous with respect to two variables $(x, u) \in J \times E$, which completes the proof.

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