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ON f-THEIN SETS

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In [1], [2] and [3] some special cases of a Turán's problem are solved. This problem can be generalized in the following way:

Let \( f : N^k \to N \) (\( N \) — the set of all positive integers), \( k \in N, k > 1 \). The set \( M \) (\( M \subset N \)) is said to be \( f \)-thin if \( f(x_1, \ldots, x_k) \notin M \) for each \( k \)-tuple of distinct numbers from \( M \). Let \( f^*(n) = \max \{ m : \{n, n + 1, \ldots, m\} \text{ can be decomposed into two } \) \( f \)-thin sets, \( \}\), provided that the function \( f^* \) exists. We shall find an upper estimate for a class of functions \( f^* \). Let us remark that e.g. for the function \( f : N^2 \to N \) defined by \( f(x_1, x_2) = x_1 + x_2 \), for \( x_1 \) and \( x_2 \) odd, and \( f(x_1, x_2) = 1 \) in the opposite case, the function \( f^* \) does not exist. Indeed, \( N \) can be decomposed into the set \( A \) of all even numbers and \( B = N - A \). \( A \) and \( B \) are infinite \( f \)-thin sets.

In the above mentioned papers additive \( k \)-thin and multiplicatively \( k \)-thin sets are investigated, i.e. functions \( a_k(x_1, \ldots, x_k) = x_1 + \ldots + x_k \) and \( m_k(x_1, \ldots, x_k) = x_1 \ldots x_k \) are considered. It is proved that \( a_k(n) \geq n(k^2 + k - 1) + \frac{3}{2}(k - 1) \).

\((k^2 + 2k - 1) - 1 \) holds for \( k > 1 \), and for \( k = 2 \) and \( k = 3 \) the inequality can be replaced by equality ([1], [3]). Further, it is known that for each \( k > 1 \) there exists a polynomial \( p_k(n) \) of the degree \( k^2 + k - 1 \) such that \( m_k^*(n) \geq p_k(n) \) (\( n = 1, 2, \ldots \)), \( \lim \inf \).

\((m_k^*(n)/n^4) - n \geq 2, \lim \sup ((m_k^*(n)/n^4) - n) \leq 4, \lim \inf ((m_k^*(n)/n^{10}) - n) \geq 10, \lim \sup ((m_k^*(n)/n^{10}) - n) \leq 13 ([1], [2]). \)

The meaning of the number \( f^*(n) \) follows from its definition: For any decomposition of the set \( \{n, n + 1, \ldots, m\} \), \( m > f^*(n) \), into two disjoint sets, in one of them the equality \( x = f(x_1, \ldots, x_k) \) with unknowns \( x, x_1, \ldots, x_k \) can be solved in such a way that \( x_i \neq x_j \) whenever \( i \neq j \).

The aim of the present article is to give an upper estimate for a class of functions \( f^* \). This will prove the existence of \( f^* \). Further, Corollary of Theorem 3 gives the affirmative answer to the question raised by B. Novák in connection with his review of [2]. Let us remark that our problem has its origin in a problem of I. Schur. This
problem and also some of its generalizations are treated in the third part of the monograph \[0\]. An ample list of references is also included in the monograph.

Let \(\circ\) be a binary operation in \(N (\circ : N \times N \to N)\), such that \((N, \circ)\) is a commutative group. In the following definitions we use \(a^x = a \circ a \circ \ldots \circ a\ \text{\(x\)}\) times, \(a^1 = a\).

**Definition 1.** Let \(p (p \geq 1), q (q \geq 1), 0 \leq c_1 < c_2 < \ldots < c_p, 0 \leq d_1 < d_2 < \ldots < d_q\) be integers, let \(\alpha_i, \beta_j\) be positive integers \((i = 1, \ldots, p; j = 1, \ldots, q)\). The binary operation \(\circ\) is said to have the property \(A\) if the assumption that the equation
\[
(a_1^{\alpha_1} \circ a_2^{\alpha_2} \circ \ldots \circ a_p^{\alpha_p}) = (b_1^{\beta_1} \circ b_2^{\beta_2} \circ \ldots \circ b_q^{\beta_q})
\]
\((a_i = n + c_i, i = 1, \ldots, p; b_j = n + d_j, j = 1, \ldots, q)\) is fulfilled for infinitely many \(n \in N\) implies \(p = q, c_i = d_i\) and \(\alpha_i = \beta_i\) for \(i = 1, \ldots, p\).

**Definition 2.** The binary operation \(\circ\) is said to have the property \(B\) if the assumption
\[
\lim_{n \to \infty} (a_1^{\alpha_1} \circ \ldots \circ a_p^{\alpha_p})/(b_1^{\beta_1} \circ \ldots \circ b_q^{\beta_q}) > 1.
\]

**Definition 3.** We shall say that a function \(f : N^p \to N\) is a quasi-polynomial of a degree \(\alpha_1 + \ldots + \alpha_p\) if \(f(x_1, \ldots, x_p) = x_1^{\alpha_1} \circ \ldots \circ x_p^{\alpha_p}\). A quasi-polynomial of a degree \(k, f(x_1, \ldots, x_k) = x_1 \circ \ldots \circ x_k\), is said to be an \(AB\)-function if the operation \(\circ\) has properties \(A\) and \(B\).

**Example 1.** Let \(s \in N\) and let the operation \(\circ\) be determined in terms of the usual multiplication by \(x \circ y = sxy\). Then the function \(m_{k,s}(x_1, \ldots, x_k) = x_1 \circ \ldots \circ x_k = s^{k-1}x_1 \ldots x_k\) is an \(AB\)-function.

Indeed, if the equality \((*)\), which has the form
\[
s^{k-1}(n + c_1)^{\alpha_1} \ldots (n + c_p)^{\alpha_p} = s^{\beta_1}(n + d_1)^{\beta_1} \ldots (n + d_q)^{\beta_q}
\]
\((\alpha = \alpha_1 + \ldots + \alpha_p, \beta = \beta_1 + \ldots + \beta_q)\), is fulfilled for infinitely many \(n\), then the properties of polynomials defined on the infinite integral domain imply \(p = q, c_i = d_i\) and \(\alpha_i = \beta_i\) for each \(i = 1, \ldots, p\). The inequality \((**)*\) is obviously fulfilled as well.

It follows from Example 1 that for each \(k > 1\) there exists infinitely many \(AB\)-functions.

**Theorem 1.** Let \(f = f(x_1, \ldots, x_k)\) be an \(AB\)-function. Then
\[(a)\] for each \(k > 1\) there exists \(n_k \in N\) such that
\[
f^*(n) < \max_{(\{\alpha_1, \ldots, \alpha_k + 2\}) = L} \{ \min_{a_i \circ (a_i^{k-1} \circ \ldots \circ a_{k+2}^{k-1})} \},
\]
where \(L = \{n, n + 1, \ldots, n + 2k + 2\},\) holds for every \(n \geq n_k;\)
(b) for each \(k > 6\) there exists \(n_k \in \mathbb{N}\) such that 
\[
f^*(n) < \max_{\{a_1, \ldots, a_{k+2}\} \subseteq M} \left\{ \min_{i=1, \ldots, k+2} \{a_i \circ (a_i^{k-1} \ldots a_{k+2}^{k-1})\} \right\},
\]
where \(M = \{n, n + 1, \ldots, n + 2k + 1\}\), holds for every \(n \geq n_k\).

Proof. First we prove part (b) of Theorem 1. Let us suppose that the set \(\{n, n + 1, \ldots, m\}\) is decomposed into two disjoint \(f\)-thin sets \(A\) and \(B\). We shall show the existence of a number \(m_n\) such that \(m_n \in A\) and \(m_n \in B\). Hence we can conclude \(f^*(n) < m_n\). Any distribution of numbers of the set \(M = \{n, n + 1, \ldots, n + 2k + 1\}\) with \(2k + 2\) elements into sets \(A' = A \cap M\) and \(B' = B \cap M\) leads to one of the following two cases: (i) each of the sets \(A'\) and \(B'\) contains \(k + 1\) elements; (ii) one of the sets \((A'\) or \(B')\) contains at least \(k + 2\) elements. Further, we shall consider a finite number of quasi-polynomials. Taking into account property \(A\) we can choose \(n_0 \in \mathbb{N}\) such that different quasi-polynomials have different values whenever their arguments are greater than \(n_0\). In the sequel we deal only with such arguments, i.e. we suppose \(n \geq n_0\).

(i) Let \(\{a_1, \ldots, a_{k+1}\} \subseteq A\) and \(\{b_1, \ldots, b_{k+1}\} \subseteq B\) \((M = \{a_1, \ldots, a_{k+1}, b_1, \ldots, b_{k+1}\})\).

Lemma. \(a_1 \circ \ldots \circ a_{k+1} \in B, b_1 \circ \ldots \circ b_{k+1} \in A\).

Proof of Lemma. Indirectly: Let us suppose \(a = a_1 \circ \ldots \circ a_{k+1} \in A\). If \(a_i \circ a_j \in A\) \((1 \leq i < j \leq k + 1)\), then \((a_i \circ a_j) \circ a_1 \circ \ldots \circ a_{i-1} \circ a_{i+1} \circ \ldots \circ a_{j-1} \circ a_{j+1} \circ \ldots \circ a_{k+1} = a \in B\) and hence \(a_i \circ a_j \in B\). Consequently
\[
(a_1 \circ a_2) \circ (a_2 \circ a_3) \circ \ldots \circ (a_k \circ a_{k+1}) = a_1 \circ a_2^2 \circ \ldots \circ a_k^2 \circ a_{k+1} \in A.
\]

On the other hand, \(a \circ a_2 \circ \ldots \circ a_k = a_1 \circ a_2^2 \circ \ldots \circ a_k^2 \circ a_{k+1} \in B\) which contradicts (1). The proof of the second part of the statement of Lemma is analogous.

Obviously \(b_1 \circ \ldots \circ b_k \in A\), and \(t_1 = a_1 \circ \ldots \circ a_{k-1} \circ (b_1 \circ \ldots \circ b_k) \in B, t_2 = a_1 \circ \ldots \circ a_{k-2} \circ a_k \circ (b_1 \circ \ldots \circ b_k) \in B\). Hence \(t_1 = t_1 \circ t_2 = (a_1 \circ \ldots \circ a_{k+1}) \circ (b_1 \circ \ldots \circ b_k) \in B\). Consequently \(w = t \circ (b_1 \circ \ldots \circ b_{k-1} \circ b_{k+1}) = a_1 \circ \ldots \circ a_{k-3} \circ a_{k-1} = a_1^4 \circ a_2^2 \circ \ldots \circ a_{k-3} \circ a_{k-1} \circ a_{k+1} \circ b_1 \circ b_2 \circ \ldots \circ b_{k-3} \circ b_{k-2} \circ b_{k+1} \in B\). If we interchange symbols “a” and “b” as well as “A” and “B” we have a proof for \(w \in A\). Hence for the given decomposition of the set \(M\), the number expressed by the quasi-polynomial \(w\) of the degree \(8k - 6\) belongs neither to \(A\) nor to \(B\).

(ii) Let us suppose \(\{a_1, \ldots, a_{k+2}\} \subseteq A\). Put \((\text{for } 1 \leq i < j \leq k + 2) u_{i,j} = a_1 \circ \ldots \circ a_{i-1} \circ a_{i+1} \circ \ldots \circ a_{j-1} \circ a_{j+1} \circ \ldots \circ a_{k+2}\). Obviously \(u_{i,j} \in B\). Hence \(u = u_{2,3} \circ u_{3,4} \circ \ldots \circ u_{k+1,k+2} = a_1^k \circ a_2^{k-1} \circ a_3^{k-2} \circ \ldots \circ a_{k-1}^{k-2} \circ a_k \circ a_{k+2}^{k-1} \in A\) and
\[
z = u \circ a_3 \circ \ldots \circ a_{k+1} = a_1^k \circ a_2^{k-1} \circ \ldots \circ a_{k+2}^{k-1} \in B.
\]
Taking into consideration the proof of Lemma we easily see that \( v_1 = a_1 \circ \ldots \circ a_{i-1} \circ a_{i+1} \circ \ldots \circ a_{k+2} \in B \) holds for each \( i = 2, \ldots, k \). Hence \( v_2 = u_k \circ u_{k+1,k+2} = a_1^k \circ a_2^{k-1} \circ \ldots \circ a_{k+2}^{k-1} \in A \). This contradicts (2). Hence for the given decomposition of the set \( M \), the number expressed by the quasi-polynomial \( z \) of the degree \( k^2 + k - 1 \) belongs neither to \( A \) nor to \( B \).

With respect to the assumption \( k > 6 \), the degree of the quasi-polynomial \( z \) is greater than that of the quasi-polynomial \( w \) as well as than those of the other quasi-polynomials \( p \) from the above considerations. It follows from the property \( B \) that there exists \( n_i \) such that \( z > w \) and \( z > p \) whenever \( n \geq n_i \). Put \( n_k = \max \{n_0, n_1\} \). The estimate for the function \( f^* \) is determined by the quasi-polynomial \( z = P_k(x_1, \ldots, x_{k+2}) = x_1^k \circ x_2^{k-1} \circ x_3^{k-1} \circ \ldots \circ x_{k+2}^{k-1} \). The above consideration has concerned any subset of \( M \) with \( k + 2 \) elements. Therefore

\[
f^*(n) \leq \max_{(a_1, \ldots, a_{k+2}) \in M} \{ \min_{(j_1, \ldots, j_{k+2}) \in M} \{ a_{j_1}^k \circ a_{j_2}^{k-1} \circ \ldots \circ a_{j_{k+2}}^{k-1} \} \},
\]

where \((j_1, \ldots, j_{k+2})\) runs over all orders of numbers \((1, \ldots, k + 2)\).

We prove part (a) of Theorem 1. Let us suppose that the set \( L = \{n, n + 1, \ldots, n + 2k + 2\} \) with \( 2k + 3 \) elements is decomposed into two disjoint \( f \)-thin sets \( A \) and \( B \). In any distribution of numbers of the set \( L \) either \( A' = A \cap L \) or \( B' = B \cap L \) contains at least \( k + 2 \) elements. Let us suppose \( \{a_1, \ldots, a_{k+2}\} \subset A \). It is obvious that the method of the proof of part (b) (ii) is applicable in this case. Since the sets \( L \) and \( M \) are different, the estimate of the function \( f^* \) for \( n \geq n_k \) (\( n_k \) is determined by conditions analogous to those from the proof of part (b)) is determined by the inequality

\[
f^*(n) \leq \max_{(a_1, \ldots, a_{k+2}) \in L} \{ \min_{(j_1, \ldots, j_{k+2}) \in L} \{ a_{j_1}^k \circ a_{j_2}^{k-1} \circ \ldots \circ a_{j_{k+2}}^{k-1} \} \},
\]

where \((j_1, \ldots, j_{k+2})\) runs over all orders of numbers \((1, \ldots, k + 2)\). This completes the proof of Theorem 1.

Let us apply Theorem 1 to the function from Example 1.

**Theorem 2.** Let \( s \in N, k > 1 \) and \( m_{k,s}(x_1, \ldots, x_k) = s^{k-1}x_1 \ldots x_k \). Then

(a) there exists \( n_k \in N \) and a polynomial \( Q_{k,s} \) of the degree \( k^2 + k - 1 \) \( (Q_{k,s}(n) = s^{k^2+k-2}(n^{k^2+k-1} + C_k n^{k^2+k-2} + \ldots), \) \( C_k = k(k+1) + \frac{1}{2}(k^2 - 1)(3k + 4) \) such that \( m^*_{k,s}(n) < Q_{k,s}(n) \) holds for each \( n \geq n_k \);

(b) for \( k > 6 \) there exists \( n_k \in N \) and a polynomial \( q_{k,s} \) of the degree \( k^2 + k - 1 \) \( (q_{k,s}(n) = s^{k^2+k-2}(n^{k^2+k-1} + D_k n^{k^2+k-2} + \ldots), \) \( D_k = k^2 + \frac{1}{2}(k^2 - 1)(3k + 2) \) such that \( m^*_{k,s}(n) < q_{k,s}(n) \) holds for each \( n \geq n_k \).

**Proof.** Theorem 2 is a consequence of Theorem 1. It is easy to see that the quasi-polynomial \( P_k \) introduced in the proof of Theorem 1 is of the form \( P_k(x_1, \ldots, x_{k+2}) = s^{k^2+k-2}x_1 x_2^{k-1} \ldots x_{k+2}^{k-1} \). Hence in the case (a),
\[
\max \left\{ \min_{(a_1, \ldots, a_{k+2}) \in L} \{s^{k+2-k-2} a_i (a_1 \ldots a_{k+2})^{k-1}\} \right\} = \\
= s^{k+2-k-2} (n + k + 1)^k (n + k + 2)^{k-1} \ldots (n + 2k + 2)^{k-1} = Q_{k,s}(n)
\]
for every sufficiently large \(n\). In the case (b),
\[
\max \left\{ \min_{(a_1, \ldots, a_{k+2}) \in M} \{s^{k+2-k-2} a_i (a_1 \ldots a_{k+2})^{k-1}\} \right\} = \\
= s^{k+2-k-2} (n + k)^k (n + k + 1)^{k-1} \ldots (n + 2k + 1)^{k-1} = q_{k,s}(n)
\]
holds for each sufficiently large \(n\).

**Theorem 3.** Let \(s \in \mathbb{N}\), \(m_{k,s}(x_1, \ldots, x_k) = s^{k-1}x_1 \ldots x_k\). Then
\[
\liminf_{n \to \infty} \left( \frac{(m_{k,s}(n)/n^{k+k-2}) - (s^{k+2-k-2}n)}{s^{k+2-k-2}(n + 2k + 1)^{k-1}} \right) \geq s^{k+2-k-2} \cdot \frac{1}{2}(k - 1) (k^2 + k - 2)
\]
and
\[
\limsup_{n \to \infty} \left( \frac{(m_{k,s}(n)/n^{k+k-2}) - (s^{k+2-k-2}n)}{s^{k+2-k-2}(n + 2k + 1)^{k-1}} \right) \leq s^{k+2-k-2}(k(k + 1) + \frac{1}{2}(k^2 - 1) (3k + 4))
\]
holds for each \(k > 1\). If \(k > 6\), then
\[
\limsup_{n \to \infty} \left( \frac{(m_{k,s}(n)/n^{k+k-2}) - (s^{k+2-k-2}n)}{s^{k+2-k-2}(n + 2k + 1)^{k-1}} \right) \leq s^{k+2-k-2}(3k + 2))
\]

**Proof.** Upper estimates of \(\limsup\) are immediate consequences of Theorem 2. If for each \(n \in \mathbb{N}\) we put \(\alpha = m_{k,s}(n, n + 1, \ldots, n + k + 1)\), \(\beta = m_{k,s}(n, n + 1, \ldots, n + k - 2, \beta)\), then it follows from the properties of multiplication that \(A = \{n, n + 1, \ldots, \alpha - 1\} \cup \{\beta, \beta + 1, \ldots, \gamma - 1\}\), \(B = \{n, n + 1, \ldots, \beta - 1\}\) provide a decomposition of the set \(\{n, n + 1, \ldots, \gamma - 1\}\) into two \(m_{k,s}\)-thin sets \(A\) and \(B\). Hence \(m_{k,s}^*(n) \geq s^{k+2-k-2}(n^{k+k-1} + \frac{1}{2}(k - 1) (k^2 + 2k - 2)n^{k+k-2} + \ldots)\) holds for each \(n \in \mathbb{N}\). The last inequality yields the lower estimate for \(\liminf_{n \to \infty} \left( \frac{(m_{k,s}(n)/n^{k+k-2}) - (s^{k+2-k-2}n)}{s^{k+2-k-2}(n + 2k + 1)^{k-1}} \right)\).

**Corollary.** Let \(s \in \mathbb{N}\) and \(k > 1\). Then
\[
m_{k,s}^*(n)/n^{k+k-2} = s^{k+2-k-2}n + O(1).
\]

**Remark.** It is easy to see that the quasi-polynomial \(m_{k,s}(x_1, \ldots, x_k) = x_1 \circ \ldots \circ x_k, s \in \mathbb{N}, t \in \mathbb{N} \cup \{0\}\), determined by the operation \(x \circ y = s(x + t)(y + t) - t\) is an \(AB\)-function. The function \(m_{k,s}\) from Example 1 is its special case, \(m_{k,s} = m_{k,s,0}\). This suggests the question: What is the general form of any \(AB\)-function?
References


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