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ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY

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ON f-THIN SETS

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In [1], [2] and [3] some special cases of a Turán's problem are solved. This problem can be generalized in the following way:

Let $f : N^k \to N$ (N – the set of all positive integers), $k \in N$, $k > 1$. The set M ($M \subset N$) is said to be f-thin if $f(x_1, \ldots, x_k) \notin M$ for each k-tuple of distinct numbers from M. Let $f^*(n) = \max \{m : \{n, n + 1, \ldots, m\}$ can be decomposed into two f-thin sets$\}$, provided that the function f^* exists. We shall find an upper estimate for a class of functions f^*. Let us remark that e.g. for the function $f : N^2 \to N$ defined by $f(x_1. x_2) = x_1 + x_2$, for x_1 and x_2 odd, and $f(x_1, x_2) = 1$ in the opposite case, the function f^* does not exist. Indeed, N can be decomposed into the set A of all even numbers and $B = N - A$. A and B are infinite f-thin sets.

In the above mentioned papers additive k-thin and multiplicatively k-thin sets are investigated, i.e. functions $a_k(x_1, \ldots, x_k) = x_1 + \ldots + x_k$ and $m_k(x_1, \ldots, x_k) = x_1 \ldots x_k$ are considered. It is proved that $a_k^*(n) \geqq n(k^2 + k - 1) + \frac{1}{2}(k - 1) . (k^2 + 2k - 2) - 1$ holds for $k > 1$, and for $k = 2$ and $k = 3$ the inequality can be replaced by equality ([1], [3]). Further, it is known that for each $k > 1$ there exists a polynomial $p_k(n)$ of the degree $k^2 + k - 1$ ($p_k(n) = n^{k^2+k-1} + \frac{1}{2}(k - 1) . (k^2 + k - 2) n^{k^2+k-2} + \ldots$) such that $m_k^*(n) \geqq p_k(n)$ ($n = 1, 2, \ldots$), $\liminf_{n\to\infty} . ((m_2^*(n)/n^4) - n) \geqq 2$, $\limsup_{n\to\infty} ((m_2^*(n)/n^4) - n) \leqq 4$, $\liminf_{n\to\infty} ((m_3^*(n)/n^{10}) - n) \geqq 10$, $\limsup_{n\to\infty} ((m_3^*(n)/n^{10}) - n) \leqq 13$ ([1], [2]).

The meaning of the number $f^*(n)$ follows from its definition: For any decomposition of the set $\{n, n + 1, \ldots, m\}$, $m > f^*(n)$, into two disjoint sets, in one of them the equality $x = f(x_1, \ldots, x_n)$ with unknowns $x, x_1, \ldots, x_k$ can be solved in such a way that $x_i \neq x_j$ whenever $i \neq j$.

The aim of the present article is to give an upper estimate for a class of functions f^*. This will prove the existence of f^*. Further, Corollary of Theorem 3 gives the affirmative answer to the question raised by B. Novák in connection with his review of [2]. Let us remark that our problem has its origin in a problem of I. Schur. This problem and also some of its generalizations are treated in the third part of the monograph [0]. An ample list of references is also included in the monograph.

Let $\circ$ be a binary operation in N ($\circ: N \times N \to N$), such that $(N, \circ)$ is a commutative group. In the following definitions we use $a^\alpha = a \circ a \circ \ldots \circ a$ α-times, $a^1 = a$.

Definition 1. Let p $(p \geqq 1)$, q $(q \geqq 1)$, $0 \leqq c_1 < c_2 < \ldots < c_p$, $0 \leqq d_1 < d_2 < \ldots \ldots < d_q$ be integers, let α_i, β_j be positive integers $(i = 1, \ldots, p;\ j = 1, \ldots, q)$. The binary operation $\circ$ is said *to have the property A* if the assumption that the equation

$$(*) \qquad a_1^{\alpha_1} \circ a_2^{\alpha_2} \circ \ldots \circ a_p^{\alpha_p} = b_1^{\beta_1} \circ b_2^{\beta_2} \circ \ldots \circ b_q^{\beta_q}$$

$(a_i = n + c_i,\ i = 1, \ldots, p;\ b_j = n + d_j,\ j = 1, \ldots, q)$ is fulfilled for infinitely many $n \in N$ implies $p = q$, $c_i = d_i$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, p$.

Definition 2. The binary operation $\circ$ is said *to have the property B* if the assumption $\alpha_1 + \ldots + \alpha_p > \beta_1 + \ldots + \beta_q$ implies

$$(**) \qquad \liminf_{n \to \infty} (a_1^{\alpha_1} \circ \ldots \circ a_p^{\alpha_p}) / (b_1^{\beta_1} \circ \ldots \circ b_q^{\beta_q}) > 1 .$$

Definition 3. We shall say that a function $f: N^p \to N$ is *a quasi-polynomial of a degree* $\alpha_1 + \ldots + \alpha_p$ if $f(x_1, \ldots, x_p) = x_1^{\alpha_1} \circ \ldots \circ x_p^{\alpha_p}$. A quasi-polynomial of a degree k, $f(x_1, \ldots, x_k) = x_1 \circ \ldots \circ x_k$, is said to be *an AB-function* if the operation $\circ$ has properties A and B.

Example 1. Let $s \in N$ and let the operation $\circ$ be determined in terms of the usual multiplication by $x \circ y = sxy$. Then the function $m_{k,s}(x_1, \ldots, x_k) = x_1 \circ \ldots \circ x_k = s^{k-1} x_1 \ldots x_k$ is an AB-function.

Indeed, if the equality $(*)$, which has the form

$$s^{\alpha-1}(n + c_1)^{\alpha_1} \ldots (n + c_p)^{\alpha_p} = s^{\beta-1}(n + d_1)^{\beta_1} \ldots (n + d_q)^{\beta_q}$$

$(\alpha = \alpha_1 + \ldots + \alpha_p,\ \beta = \beta_1 + \ldots + \beta_q)$, is fulfilled for infinitely many n, then the properties of polynomials defined on the infinite integral domain imply $p = q$, $c_i = d_i$ and $\alpha_i = \beta_i$ for each $i = 1, \ldots, p$. The inequality $(**)$ is obviously fulfilled as well.

It follows from Example 1 that for each $k > 1$ there exists infinitely many AB-functions.

Theorem 1. *Let* $f = f(x_1, \ldots, x_k)$ *be an AB-function. Then*

(a) *for each* $k > 1$ *there exists* $n_k \in N$ *such that*

$$f^*(n) < \max_{\{a_1, \ldots, a_{k+2}\} \subset L} \{ \min_{i=1,\ldots,k+2} \{a_i \circ (a_1^{k-1} \circ \ldots \circ a_{k+2}^{k-1})\}\} ,$$

where $L = \{n, n + 1, \ldots, n + 2k + 2\}$, *holds for every* $n \geqq n_k$;

(b) *for each* $k > 6$ *there exists* $n_k \in N$ *such that*

$$f^*(n) < \max_{\{a_1,\dots,a_{k+2}\} \subset M} \{ \min_{i=1,\dots,k+2} \{a_i \circ (a_1^{k-1} \dots a_{k+2}^{k-1})\}\} ,$$

where $M = \{n, n+1, \dots, n+2k+1\}$, *holds for every* $n \geqq n_k$.

Proof. First we prove part (b) of Theorem 1. Let us suppose that the set $\{n, n+1, \dots, m\}$ is decomposed into two disjoint f-thin sets A and B. We shall show the existence of a number m_n such that $m_n \in A$ and $m_n \in B$. Hence we can conclude $f^*(n) < m_n$, Any distribution of numbers of the set $M = \{n, n+1, \dots, n+2k+1\}$ with $2k+2$ elements into sets $A' = A \cap M$ and $B' = B \cap M$ leads to one of the following two cases: (i) each of the sets A' and B' contains $k+1$ elements; (ii) one of the sets (A' or B') contains at least $k+2$ elements. Further, we shall consider a finite number of quasi-polynomials. Taking into account property A we can choose $n_0 \in N$ such that different quasi-polynomials have different values whenever their arguments are greater than n_0. In the sequel we deal only with such arguments, i.e. we suppose $n \geqq n_0$.

(i) Let $\{a_1, \dots, a_{k+1}\} \subset A$ and $\{b_1, \dots, b_{k+1}\} \subset B$ $(M = \{a_1, \dots, a_{k+1}, b_1, \dots \dots, b_{k+1}\})$.

Lemma. $a_1 \circ \dots \circ a_{k+1} \in B$, $b_1 \circ \dots \circ b_{k+1} \in A$.

Proof of Lemma. Indirectly: Let us suppose $a = a_1 \circ \dots \circ a_{k+1} \in A$. If $a_i \circ a_j \in A$ $(1 \leqq i < j \leqq k+1)$, then $(a_i \circ a_j) \circ a_1 \circ \dots \circ a_{i-1} \circ a_{i+1} \circ \dots \circ a_{j-1} \circ a_{j+1} \circ \dots \circ a_{k+1} = a \in B$ and hence $a_i \circ a_j \in B$. Consequently

$$(1) \qquad (a_1 \circ a_2) \circ (a_2 \circ a_3) \circ \dots \circ (a_k \circ a_{k+1}) = a_1 \circ a_2^2 \circ \dots \circ a_k^2 \circ a_{k+1} \in A .$$

On the other hand, $a \circ a_2 \circ \dots \circ a_k = a_1 \circ a_2^2 \circ \dots \circ a_k^2 \circ a_{k+1} \in B$ which contradicts (1). The proof of the second part of the statement of Lemma is analogous.

Obviously $b_1 \circ \dots \circ b_k \in A$, and $t_1 = a_1 \circ \dots \circ a_{k-1} \circ (b_1 \circ \dots \circ b_k) \in B$, $t_2 = a_1 \circ \dots \circ a_{k-2} \circ a_k \circ (b_1 \circ \dots \circ b_k) \in B$. Hence $t = t_1 \circ t_2 \circ (a_1 \circ \dots \circ a_{k+1}) \circ b_1 \circ \dots \circ b_{k-3} = a_1^3 \circ a_2^3 \circ \dots \circ a_{k-2}^3 \circ a_{k-1}^2 \circ a_k^2 \circ a_{k+1} \circ b_1^3 \circ b_2^3 \circ \dots \circ b_{k-3}^3 \circ b_{k-2}^2 \circ b_{k-1}^2 \circ b_k^2 \in A$. Consequently $w = t \circ (b_1 \circ \dots \circ b_{k-1} \circ b_{k+1}) \circ a_1 \circ \dots \circ a_{k-3} \circ a_{k-1} = a_1^4 \circ a_2^4 \circ \dots \circ a_{k-3}^4 \circ a_{k-2}^3 \circ a_{k-1}^3 \circ a_k^2 \circ a_{k+1} \circ b_1^4 \circ b_2^4 \circ \dots \circ b_{k-3}^4 \circ b_{k-2}^3 \circ b_{k-1}^3 \circ b_k^2 \circ b_{k+1} \in B$. If we interchange symbols "a" and "b" as well as "A" and "B" we have a proof for $w \in A$. Hence for the given decomposition of the set M, the number expressed by the quasi-polynomial w of the degree $8k - 6$ belongs neither to A nor to B.

(ii) Let us suppose $\{a_1, \dots, a_{k+2}\} \subset A$. Put (for $1 \leqq i < j \leqq k+2$) $u_{i,j} = a_1 \circ \dots \circ a_{i-1} \circ a_{i+1} \circ \dots \circ a_{j-1} \circ a_{j+1} \circ \dots \circ a_{k+2}$. Obviously $u_{i,j} \in B$. Hence $u = u_{2,3} \circ u_{3,4} \circ \dots \circ u_{k+1,k+2} = a_1^k \circ a_2^{k-1} \circ a_3^{k-2} \circ \dots \circ a_{k+1}^{k-2} \circ a_{k+2}^{k-1} \in A$ and

$$(2) \qquad z = u \circ a_3 \circ \dots \circ a_{k+1} = a_1^k \circ a_2^{k-1} \circ \dots \circ a_{k+2}^{k-1} \in B \cdot$$

Taking into consideration the proof of Lemma we easily see that $v_i = a_1 \circ \ldots \circ a_{i-1} \circ \circ a_{i+1} \circ \ldots \circ a_{k+2} \in B$ holds for each $i = 2, \ldots, k$. Hence $v_2 \circ \ldots \circ v_k \circ u_{k+1,k+2} = a_1^k \circ a_2^{k-1} \circ \ldots \circ a_{k+2}^{k-1} \in A$. This contradicts (2). Hence for the given decomposition of the set M, the number expressed by the quasi-polynomial z of the degree $k^2 + k - 1$ belongs neither to A nor to B.

With respect to the assumption $k > 6$, the degree of the quasi-polynomial z is greater than that of the quasi-polynomial w as well as than those of the other quasi-polynomials p from the above considerations. It follows from the property B that there exists n_1 such that $z > w$ and $z > p$ whenever $n \geqq n_1$. Put $n_k = \max \{n_0, n_1\}$. The estimate for the function f^* is determined by the quasi-polynomial $z = P_k(x_1, \ldots, x_{k+2}) = x_1^k \circ x_2^{k-1} \circ x_3^{k-1} \circ \ldots \circ x_{k+2}^{k-1}$. The above consideration has concerned any subset of M with $k + 2$ elements. Therefore

$$f^*(n) < \max_{\{a_1,\ldots,a_{k+2}\} \subset M} \{ \min_{(j_1,\ldots,j_{k+2})} \{a_{j_1}^k \circ a_{j_2}^{k-1} \circ \ldots \circ a_{j_{k+2}}^{k-1}\}\} ,$$

where $(j_1, \ldots, j_{k+2})$ runs over all orders of numbers $(1, \ldots, k + 2)$.

We prove part (a) of Theorem 1. Let us suppose that the set $L = \{n, n + 1, \ldots \ldots, n + 2k + 2\}$ with $2k + 3$ elements is decomposed into two disjoint f-thin sets A and B. In any distibution of numbers of the set L either $A' = A \cap L$ or $B' = B \cap L$ contains at least $k + 2$ elements. Let us suppose $\{a_1, \ldots, a_{k+2}\} \subset A$. It is obvious that the method of the proof of part (b) (ii) is applicable in this case. Since the sets L and M are different, the estimate of the function f^* for $n \geqq n_k$ (n_k is determined by conditions analogous to those from the proof of part (b)) is determined by the inequality

$$f^*(n) < \max_{\{a_1,\ldots,a_{k+2}\} \subset L} \{ \min_{(j_1,\ldots,j_{k+2})} \{a_{j_1}^k \circ a_{j_2}^{k-1} \circ \ldots \circ a_{j_{k+2}}^{k-1}\}\} ,$$

where $(j_1, \ldots, j_{k+2})$ runs over all orders of numbers $(1, \ldots, k + 2)$. This completes the proof of Theorem 1.

Let us apply Theorem 1 to the function from Example 1.

Theorem 2. *Let $s \in N$, $k > 1$ and $m_{k,s}(x_1, \ldots, x_k) = s^{k-1}x_1 \ldots x_k$. Then*

(a) *there exists $n_k \in N$ and a polynomial $Q_{k,s}$ of the degree $k^2 + k - 1$ ($Q_{k,s}(n) = s^{k^2+k-2}(n^{k^2+k-1} + C_k n^{k^2+k-2} + \ldots)$, $C_k = k(k + 1) + \frac{1}{2}(k^2 - 1)(3k + 4)$) such that $m_{k,s}^*(n) < Q_{k,s}(n)$ holds for every $n \geqq n_k$;*

(b) *for $k > 6$ there exists $n_k \in N$ and a polynomial $q_{k,s}$ of the degree $k^2 + k - 1$ ($q_{k,s}(n) = s^{k^2+k-2}(n^{k^2+k-1} + D_k n^{k^2+k-2} + \ldots)$, $D_k = k^2 + \frac{1}{2}(k^2 - 1)(3k + 2)$) such that $m_{k,s}^*(n) < q_{k,s}(n)$ holds for each $n \geqq n_k$.*

Proof. Theorem 2 is a consequence of Theorem 1. It is easy to see that the quasi-polynomial P_k introduced in the proof of Theorem 1 is of the form $P_k(x_1, \ldots, x_{k+2}) = s^{k^2+k-2}x_1^k x_2^{k-1} \ldots x_{k+2}^{k-1}$. Hence in the case (a),

$$\max_{\{a_1,\dots,a_{k+2}\}\subset L} \{ \min_{i=1,\dots,k+2} \{s^{k^2+k-2} a_i (a_1 \dots a_{k+2})^{k-1}\}\} =$$
$$= s^{k^2+k-2}(n + k + 1)^k (n + k + 2)^{k-1} \dots (n + 2k + 2)^{k-1} = Q_{k,s}(n)$$

for every sufficiently large n. In the case (b),

$$\max_{\{a_1,\dots,a_{k+2}\}\subset M} \{ \min_{i=1,\dots,k+2} \{s^{k^2+k-2} a_i (a_1 \dots a_{k+2})^{k-1}\}\} =$$
$$= s^{k^2+k-2}(n + k)^k (n + k + 1)^{k-1} \dots (n + 2k + 1)^{k-1} = q_{k,s}(n)$$

holds for each sufficiently large n.

Theorem 3. *Let $s \in N$, $m_{k,s}(x_1, \dots, x_k) = s^{k-1} x_1 \dots x_k$. Then*

$$\liminf_{n\to\infty} ((m^*_{k,s}(n)/n^{k^2+k-2}) - (s^{k^2+k-2} n)) \geqq s^{k^2+k-2} \cdot \tfrac{1}{2}(k - 1)(k^2 + k - 2)$$

and

$$\limsup_{n\to\infty} (m^*_{k,s}(n)/n^{k^2+k-2}) - (s^{k^2+k-2} n)) \leqq s^{k^2+k-2}(k(k + 1) + \tfrac{1}{2}(k^2 - 1)(3k + 4))$$

holds for each $k > 1$. If $k > 6$, then

$$\limsup_{n\to\infty} ((m^*_{k,s}(n)/n^{k^2+k-2}) - (s^{k^2+k-2} n)) \leqq s^{k^2+k-2}(k^2 + \tfrac{1}{2}(k^2 - 1)(3k + 2)).$$

Proof. Upper estimates of $\limsup_{n\to\infty} ((m^*_{k,s}(n)/n^{k^2+k-2}) - (s^{k^2+k-2} n))$ are immediate consequences of Theorem 2. If for each $n \in N$ we put $\alpha = m_{k,s}(n, n + 1, \dots, n + k + 1)$, $\beta = m_{k,s}(\alpha, \alpha + 1, \dots, \alpha + k - 1)$ and $\gamma = m_{k,s}(n, n + 1, \dots, n + k - 2, \beta)$, then it follows from the properties of multiplication that $A = \{n, n + 1, \dots \dots, \alpha - 1\} \cup \{\beta, \beta + 1, \dots, \gamma - 1\}$, $B = \{\alpha, \alpha + 1, \dots, \beta - 1\}$ provide a decomposition of the set $\{n, n + 1, \dots, \gamma - 1\}$ into two $m_{k,s}$-thin sets A and B. Hence $m^*_{k,s}(n) \geqq \gamma - 1 = s^{k^2+k-2}(n^{k^2+k-1} + \tfrac{1}{2}(k - 1)(k^2 + k - 2) n^{k^2+k-2} + \dots)$ holds for each $n \in N$. The last inequality yields the lower estimate for $\liminf_{n\to\infty} ((m^*_{k,s}(n)/n^{k^2+k-2}) - (s^{k^2+k-2} n))$.

Corollary. *Let $s \in N$ and $k > 1$. Then*

$$m^*_{k,s}(n)/n^{k^2+k-2} = s^{k^2+k-2} n + \Omega(1).$$

Remark. It is easy to see that the quasi-polynomial $m_{k,s,t}(x_1, \dots, x_k) = x_1 \circ \dots \circ x_k$, $s \in N$, $t \in N \cup \{0\}$, determined by the operation $x \circ y = s(x + t)(y + t) - t$ is an AB-function. The function $m_{k,s}$ from Example 1 is its special case, $m_{k,s} = m_{k,s,0}$. This suggests the question: What is the general form of any AB-function?

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