E. M. Ibrahim

Invariant theory under restricted groups

Časopis pro pěstování matematiky, Vol. 95 (1970), No. 4, 356--359

Persistent URL: http://dml.cz/dmlcz/108336

Časopis pro pěstování matematiky, roč. 95 (1970), Praha

INVARIANT THEORY UNDER RESTRICTED GROUPS

E. M. IBRAHIM, Cairo

(Received November 26, 1968)

The relation that expresses Schur functions in terms of the characters of the orthogonal group is[1])

I $$\{\lambda\} = [\lambda] + \sum \Gamma_{\delta\mu\lambda}[\mu]$$

where $\Gamma_{\delta\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\delta\}\,\{\mu\}$ the summation is taken w.r.t. all partitions (δ) into even parts only.

The formula that expresses the orthogonal group characters in terms of S functions is[1])

II $$[\lambda] = \{\lambda\} + \sum(-1)^{p/2}\,\Gamma_{\nu\mu\lambda}\{\mu\}$$

where $\Gamma_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\,\{\mu\}$ and (ν) is a partition of p, summed for all partitions which in Frobenius' nomenclature are of one of the forms

$$\begin{pmatrix} r+1 \\ r \end{pmatrix}, \quad \begin{pmatrix} r+1 & s+1 \\ r & s \end{pmatrix}, \quad \begin{pmatrix} r+1 & s+1 & t+1 \\ r & s & t \end{pmatrix}, \quad \ldots$$

These partitions appear in the series

$$1 - \{2\} + \{31\} - \{41^2\} - \{3^2\} - \{4^2 2\} + \ldots$$

Schur functions are given in terms of the characters of the symplectic group by the relation[2])

III $$\{\lambda\} = \langle\lambda\rangle + \sum \Gamma_{\beta\mu\lambda}\langle\mu\rangle$$

where $\Gamma_{\beta\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\beta\}\,\{\mu\}$ and $\{\beta\}$ is summed for all partitions in which each part is repeated an even number of times i.e.

$$1 + \{1^2\} + \{2^2\} + \{1^4\} + \{3^2\} + \{2^2 1^2\} + \{1^6\} + \ldots$$

[1]) Littlewood [5].

[2]) Littlewood [5].

To express the group characters of the symplectic group in terms of S functions we have[2])

IV $$\langle\lambda\rangle = \{\lambda\} + \sum(-1)^{p/2}\,\Gamma_{\alpha\mu\lambda}\{\mu\}$$

where $\Gamma_{\alpha\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\mu\}\,\{\alpha\}$, (α) is a partition of p which in Frobenius' nomenclature is of one of the forms

$$\begin{pmatrix} r \\ r+1 \end{pmatrix}, \begin{pmatrix} r & s \\ r+1 & s+1 \end{pmatrix}, \begin{pmatrix} r & s & t \\ r+1 & s+1 & t+1 \end{pmatrix}, \ldots$$

which appear in the series $1 - \{1^2\} + \{21^2\} - \{2^3\} + \{32^21\} + \ldots$

Just as the case of the full linear group of transformations the main problem is to express $[\mu] \otimes \{\lambda\}$ or $\langle\mu\rangle \otimes \{\lambda\}$ as the sum of simple characters. To evaluate these plethysms LITTLEWOOD expressed the characters $[\mu]$, $\langle\mu\rangle$ in terms of S functions by II & IV then after expansion, he expressed the S functions back into orthogonal and symplectic group characters by I & III. Use is made of the formula

$$(A - B) \otimes \{\lambda\} = A \otimes \{\lambda\} + \sum(-1)^b\,\Gamma_{\alpha\beta\lambda}(A \otimes \{\alpha\})\,(B \otimes \{\beta^*\})$$

where (β) is a partition of b, β^* is the conjugate partition & $\Gamma_{\alpha\beta\lambda}$ is the coefficient of $\{\lambda\}$ in $\{\alpha\}\,\{\beta\}$.

Later the author[3]) gave a proof to a theorem mentioned by MURNAGHAN, that, if (λ) is a partition of an even integer m then

Va $$\langle\lambda\rangle \otimes \{\mu\} = ([\lambda^*] \otimes \{\mu\})^*\,.$$

While if (λ) is a partition of odd integer

Vb $$\langle\lambda\rangle \otimes \{\mu\} = ([\lambda^*] \otimes \{\mu^*\})^*$$

which give the analyses of $\langle\lambda\rangle \otimes \{\mu\}$ when $[\lambda^*] \otimes \{\mu\}$ and $[\lambda^*] \otimes \{\mu^*\}$ are known. Also it has been proved[4]) that

VI $$[\lambda] = \langle\lambda\rangle + \Gamma_{\eta\mu\lambda}\langle\mu\rangle$$

where $\Gamma_{\eta\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\eta\}\,\{\mu\}$, $\{\eta\}$ is summed for all partitions given by the series

$$1 - \{2\} + \{1^2\} + \{2^2\} - \{21^2\} - \{2^3\} \ldots$$

and that $\sum\langle\lambda\rangle = (\sum[\lambda^*])^*$

Later Littlewood[5]) proved that

VII $$\langle\lambda\rangle = [\lambda] + \sum\Gamma_{\xi\mu\lambda}[\mu] - \sum\Gamma_{\eta\mu\lambda}[\mu]$$

[3]) Ibrahim [1].
[4]) Ibrahim [1].
[5]) Littlewood [6].

where $\Gamma_{\xi\mu\lambda}$, $\Gamma_{\eta\mu\lambda}$ are the coefficients of $\{\lambda\}$ in the product $\{\xi\}\{\mu\}$ or $\{\eta\}\{\mu\}$ respectively where ξ is summed for all partitions into not more than two even parts & η for all partitions into exactly two odd parts.

In this paper two new theorems are given:

Theorem I: *The product of the symbolic expression for a concomitant of degree n in the coefficients of a ground form of type* $[\lambda]$ *under the restricted orthogonal group of transformations by the symbolic expression of a concomitant of degree n in the coefficients of a form of type* $[\mu]$ *under the orthogonal group gives the symbolic expression of concomitants of degree n in the coefficients of a ground form of type* $[\lambda + \mu]$.

Proof. Let G & H be the symbolic expression of two forms of type $[\lambda_1, \ldots, \lambda_r]$ & & $[\mu_1, \ldots, \mu_r]$ respectively under the orthogonal group of transformations. If the same symbols are used in the two expressions then $F = GH$ may be considered as the symbolic expression for a form of type $[\lambda_1 + \mu_1, \ldots, \lambda_r + \mu_r]$. Let ξ & ζ be symbolic expression of concomitants of degree n in G and n in H. If the same symbols are used in each expression then $\xi\zeta$ will give the symbolic expression for a concomitant of degree n in F. The existence of this concomitant proves the theorem.

In terms of S functions & group characters under the orthogonal group of transformations, $[\lambda] \otimes \{n\}$ gives the concomitants of degree n in the coefficients of a ground form of type $[\lambda]$ & $[\mu] \otimes \{n\}$ gives concomitants of degree n in the coefficients of a ground form of type $[\mu]$. Then the principal parts of the products of individual terms in the expansion of $([\lambda] \otimes \{n\})([\mu] \otimes \{n\})$ appear as terms in $[\lambda + \mu] \otimes \{n\}$.

The theorem does not mean that frequency of occurrence of a partition in $[\lambda_1 + \mu_1, \ldots, \lambda_r + \mu_r] \otimes \{n\}$ is at least as great as the number of ways in which it appears as principal part of products of terms in $([\lambda] \otimes \{n\})([\mu] \otimes \{n\})$.

Example.

$$\begin{aligned}[4] \otimes \{2\} &= (\{4\} - \{2\}) \otimes \{2\} = \{4\} \otimes \{2\} - \{4\}\{2\} + \{2\} \otimes \{1^2\} = \\ &= \{8\} + \{62\} + \{4^2\} - \{6\} - \{51\} - \{42\} + \{31\} = \\ &= [8] + [62] + [6] + [42] + [4^2] + [4] + [2^2] + 2[2] + [0].\end{aligned}$$

Also $[2] \otimes \{2\} = [4] + [2^2] + [2] + [0]$.

The principal parts of the product of terms in $([2] \otimes \{2\})([2] \otimes \{2\})$ which are

$$[8] + [62] + [6] + [4] + [4^2] + [42] + [2^2] + [2] + [0]$$

appears as terms in $[4] \otimes \{2\}$.

A coefficient greater than one can be assumed when this coefficient appear in the individual terms of $[\lambda] \otimes \{n\}$ or $[\mu] \otimes \{n\}$.

Theorem II. *The product of the symbolic expression of a concomitant of degree n in the coefficients of a form of type $\langle\lambda\rangle$ under the symplectic group of transformations by the symbolic expression of a concomitant of degree n in the coefficients of a form of type $\langle\mu\rangle$ gives the symbolic expression of a concomitant of degree n in the coefficients of a form of type $\langle\lambda + \mu\rangle$.*

The proof follows as in theorem I.

Example.

$$\langle 4\rangle \otimes \{2\} = \langle 8\rangle + \langle 62\rangle + \langle 51\rangle + \langle 4\rangle + \langle 4^2\rangle + \langle 3^2\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle$$

$$\langle 2\rangle \otimes \{2\} = \langle 4\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle .$$

The principal parts of the product of terms in $(\langle 2\rangle \otimes \{2\})(\langle 2\rangle \otimes \{2\})$ which are $\langle 8\rangle + \langle 62\rangle + \langle 51\rangle + \langle 4\rangle + \langle 4^2\rangle + \langle 3^2\rangle + \langle 2^2\rangle + \langle 1^2\rangle + \langle 0\rangle$ appear as terms in $\langle 4\rangle \otimes \{2\}$.

In fact other results could be deduced as those given under the full linear group of transformations[6]).

References

[1] *Ibrahim E. M.:* On a theorem by Murnaghan. Proc. Nat. Acad. Sci. Vol. *40* No 5 1954.

[2] *Ibrahim E. M.:* The plethysm of *S*-functions. Quart. J. Math. Oxford (2) 1952.

[3] *Ibrahim E. M.:* On D. E. Littlewood algebra of *S*-function. Proc. Amer. Math. Soc. Vol. *7* No 2 1956.

[4] *Littlewood D. E.:* The theory of group characters. Oxford 1950.

[5] *Littlewood D. E.:* On invariant theory under restricted groups. Phil. Trans. Roy. Soc. Lond. Ser; A No 809 1944.

[6] *Littlewood D. E.:* On orthogonal and symplectic group characters. J. Lond. Math. Soc. *729* 1955.

Author's address: Faculty of Engineering, Ain Shams University, Abbassia, Cairo, U.A.R.

[6]) Ibrahim [2], [3].