Bohdan Zelinka
Contraction distance between isomorphism classes of graphs

Časopis pro pěstování matematiky, Vol. 115 (1990), No. 2, 211--216

Persistent URL: http://dml.cz/dmlcz/108363
CONTRACTION DISTANCE BETWEEN ISOMORPHISM CLASSES OF GRAPHS

BOHDAN ZELINKA, Liberec
(Received September 6, 1988)

Summary. A distance between isomorphism classes of connected graphs with a given number $n$ of vertices is introduced. Its definition is based on the maximum number of vertices of a graph onto which two given graphs can be contracted. The properties of this distance are studied; they are compared with the properties of other distances between isomorphism classes of graphs.

Keywords: contraction of a graph, distance, Hadwiger number, tree.

AMS classification: 05C35, 05C05.

An isomorphism class of graphs is the class of all graphs which are isomorphic to a given graph. Various distances between isomorphism classes have been introduced.

The subgraph distance $\delta$ was introduced in [6]. The distance $\delta(\mathcal{G}_1, \mathcal{G}_2)$ of two isomorphism classes $\mathcal{G}_1, \mathcal{G}_2$ of graphs with a given number $n$ of vertices is equal to $n$ minus the maximum number of vertices of a graph which is isomorphic simultaneously to an induced subgraph of a graph from $\mathcal{G}_1$ and to an induced subgraph of a graph from $\mathcal{G}_2$. This distance was modified by F. Kaden [3] and F. Sobik [5] for pairs of isomorphism classes of graphs which need not have the same number of vertices. For trees this distance was modified in [7]; instead of an induced subgraph a subtree is taken. This distance is denoted by $\delta_T$. The edge distance was introduced by V. Baláž, J. Koča, V. Kvasnička and M. Sekanina [1]. Let $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$, let $V_1, V_2$ be the vertex sets and $E_1, E_2$ the edge sets of $G_1, G_2$, respectively. Let $E_{12}$ be the edge set of a graph which is isomorphic to subgraphs of both $G_1, G_2$ and has the maximum number of edges. Then the edge distance is $\delta_E(\mathcal{G}_1, \mathcal{G}_2) = |E_1| + |E_2| - 2|E_{12}| + ||V_1| - |V_2||$. The edge rotation distance was introduced by G. Chartrand, F. Saba and H.-B. Zou [2]. If $u, v, w$ are three vertices of a graph $G$ such that $u$ is adjacent to $v$ and not to $w$, then to perform the rotation of the edge $uv$ to the position $uw$ means to delete the edge $uv$ from $G$ and to add the edge $uw$ to $G$. The edge rotation distance $\delta_R(\mathcal{G}_1, \mathcal{G}_2)$ is the minimum number of edge rotations necessary for transforming a graph from $\mathcal{G}_1$ into a graph from $\mathcal{G}_2$.

Various distances were compared in [8]. There are also other distances between isomorphism classes of graphs.

Here we shall introduce a new distance between isomorphism classes of graphs called the contraction distance. All graphs considered are finite connected undirected graphs without loops and multiple edges.
Let $G$ be a connected graph, let $\mathcal{P}$ be a partition of the vertex set $V(G)$ of $G$ with the property that each class of $\mathcal{P}$ induces a connected subgraph of $G$. Let $G'$ be the graph whose vertex set is $\mathcal{P}$ and in which two vertices $P_1, P_2$ are adjacent if and only if at least one vertex of $P_1$ is adjacent to at least one vertex of $P_2$ in $G$. Then we say that $G'$ (and every graph isomorphic to $G'$) is a contract of $G$ and was obtained from $G$ by a contraction.

If one of the classes of $\mathcal{P}$ has two elements and all the others have one element each, we say that $G'$ was obtained from $G$ by an elementary contraction.

It is easy to see that any contraction is a superposition of elementary contractions. The number of vertices of $G$ is equal to the sum of the number of vertices of $G'$ and the number of elementary contractions by whose superposition $G'$ was obtained from $G$.

Let $\mathcal{C}_n$ be the set of all isomorphism classes of connected graphs with $n$ vertices. Let $G_1 \in \mathcal{C}_n$, $G_2 \in \mathcal{C}_n$. Let $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ and let $k$ be the maximum number of vertices of a common contract $G_{12}$ of $G_1$ and $G_2$. Then $\delta_c(G_1, G_2) = n - k$ is called the contraction distance between $G_1$ and $G_2$.

In other words, $\delta_c(G_1, G_2)$ is the minimum number of elementary contractions whose superposition is a contraction of $G_1$ or of $G_2$ onto $G_{12}$.

Now we shall define an auxiliary concept. Let $G'$ be a contract of $G$, let $e$ be an edge of $G$ and let $e'$ be an edge of $G'$. We say that $e$ and $e'$ correspond to each other, if the end vertices of $e$ belong to the classes of $\mathcal{P}$ which are end vertices of $e'$.

**Theorem 1.** The functional $\delta_c$ is a metric on $\mathcal{C}_n$.

**Proof.** Every two connected graphs with $n$ vertices have a common contract, namely the graph consisting of one vertex; if $n \geq 2$, then also the complete graph with two vertices. Therefore $\delta_c(G_1, G_2)$ is well-defined. It is easy to prove that $\delta_c(G_1, G_2) \geq 0$, that $\delta_c(G_1, G_2) = 0$ if and only if $G_1 = G_2$ and that $\delta_c(G_1, G_2) = \delta_c(G_2, G_1)$. We shall prove the triangle inequality. Let $G_1, G_2, G_3$ be three isomorphism classes of connected graphs with $n$ vertices, let $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$, $G_3 \in \mathcal{G}_3$. Let $G_{12}$ (or $G_{23}$) be a common contract of $G_1$ (or $G_3$ respectively) and $G_2$ with the maximum number of vertices. The graph $G_{12}$ is obtained from $G_1$ by $\delta_c(G_1, G_2)$ elementary contractions, the graph $G_{23}$ is obtained from $G_2$ by $\delta_c(G_2, G_3)$ elementary contractions. Let $F$ be the set of edges of $G_2$ whose contractions yield $G_{23}$, let $F_0$ be the set of edges of $G_{12}$ corresponding to the edges of $F$. There may be an edge of $F$ to which no edge of $F_0$ corresponds, because this edge is contracted when forming $G_{12}$; on the other hand, obviously there may be more than one edge of $F$ which correspond to the same edge of $F_0$. Therefore $|E_0| \leq |F| = \delta_c(G_2, G_3)$. Let $G_{123}$ be the graph obtained from $G_{12}$ by the superposition of elementary contractions determined by edges of $F_0$. As $G_{12}$ is a common contract of $G_1$ and $G_2$, so is $G_{123}$. As the superposition of contractions is evidently commutative, the graph $G_{123}$ can be obtained from $G_2$ also by contracting the edges of $F$ to obtain $G_{23}$ and the edges corresponding in $G_{23}$ to the edges which were contracted in $G_2$ to obtain...
Thus $G_{123}$ is also a contract of $G_{23}$ and hence of $G_3$. It is a common contract of $G_1$ and $G_3$ and has $n - \delta_c(\mathcal{G}_1, \mathcal{G}_2) - |F_0|$ vertices. Thus $\delta_c(\mathcal{G}_1, \mathcal{G}_2) \leq \delta_c(\mathcal{G}_1, \mathcal{G}_2) + |F_0| \leq \delta_c(\mathcal{G}_1, \mathcal{G}_2) + \delta_c(\mathcal{G}_2, \mathcal{G}_3)$, which was to be proved. □

In the sequel, instead of speaking about the distance between isomorphism classes of graphs we will speak about the distance between graphs (for the sake of brevity). We will also use the corresponding notation $\delta_c(G_1, G_2)$.

The concept of contraction distance is related to the Hadwiger number (in another terminology, contraction number) which is dealt with e.g. in [4], [9]. The Hadwiger number $\eta(G)$ is the maximum number of vertices of a complete graph which is a contract of $G$.

**Theorem 2.** Let $G_1$, $G_2$ be two connected graphs with $n$ vertices. Then $\delta_c(G_1, G_2) \leq n - \min (\eta(G_1), \eta(G_2))$.

**Proof.** Evidently the complete graph with $\min (\eta(G_1), \eta(G_2))$ vertices is a common contract of $G_1$ and $G_2$. This implies the assertion. □

Now we shall compare $\delta_c(G_1, G_2)$ with $\delta(G_1, G_2)$.

**Theorem 3.** For each $n \geq 6$ there exist graphs $G_1$, $G_2$ with $n$ vertices such that $\delta_c(G_1, G_2) = [(n - 1)/2] - 1$, while $\delta(G_1, G_2) = 1$.

**Proof.** Consider a circuit $C$ with $n - 1$ vertices $u_1, \ldots, u_{n-1}$ and edges $u_iu_{i+1}$ for $i = 1, \ldots, n - 1$, the sum $i + 1$ being taken modulo $n - 1$. The graph $G_1$ (or $G_2$) is obtained from $C$ by adding a new vertex $v$ and joining it by edges with the vertices $u_1$ and $u_2$ (or $u_{k+1}$, respectively, where $k = [(n - 1)/2]$). The circuit $C$ is a common induced subgraph of $G_1$ and $G_2$ with $n - 1$ vertices and thus $\delta(G_1, G_2) = n - (n - 1) = 1$. Each of the graphs $G_1$, $G_2$ has exactly two vertices of degree 3 and consists of three paths connecting these two vertices. In $G_1$ the lengths of these paths are 1, 2 and $n - 2$, in $G_2$ they are 2, $[(n - 1)/2]$, $[(n - 1)/2]$. Hence in no contract of $G_2$ there may exist a path longer than $[(n - 1)/2]$ and connecting two vertices of degree 3. Therefore a common contract of $G_1$ and $G_2$ with the maximum number of vertices has two vertices of degree 3 and consists of three paths connecting them; their lengths are 1, 2 and $[(n - 1)/2]$. Such a graph has $[(n - 1)/2] + 2$ vertices, hence $\delta_c(G_1, G_2) = n - ([(n - 1)/2] + 2) = [(n - 1)/2] - 1$. □

**Theorem 4.** For each positive integer $p$ there exist graphs $G_1$, $G_2$ such that $\delta_c(G_1, G_2) = p$, $\delta(G_1, G_2) = 2p$.

**Proof.** The vertex set of $G_1$ is the union of disjoint sets $U = \{u_1, \ldots, u_p\}$, $V = \{v_1, \ldots, v_q\}$, $X = \{x_1, \ldots, x_a\}$, $Y = \{y_1, \ldots, y_b\}$, where $q \geq p$. Any two vertices of $U$ and any two vertices of $V$ are joined by an edge. Each vertex of $U$ with each vertex of $X$ and each vertex of $V$ with each vertex of $Y$ is joined by an edge. Further
The vertex set of $G_2$ is the union of disjoint sets $U = \{u_1, \ldots, u_p\}$, $W = \{w_1, \ldots, w_{p+2q}\}$. Any two vertices of $U$ are joined by an edge and each vertex of $U$ with each vertex of $W$ is also joined by an edge. No other edges are in $G_2$. If we contract all edges $u_iw_{i+1}$ for $i = 1, \ldots, p$ in $G_1$ and all edges $u_1w_{2q+i}$ for $i = 1, \ldots, p$ in $G_2$, we obtain isomorphic graphs. This is a minimum number of edges by whose contraction this may be done, because $G_1$ contains two disjoint cliques with $p$ vertices, while $G_2$ only one. Therefore $\delta(G_1, G_2) = p$. The largest graph isomorphic to isomorphic subgraphs of both $G_1$ and $G_2$ is evidently the graph consisting of $2q$ isolated vertices. As the number of vertices of $G_1$ as well as of $G_2$ is $2p + 2q$, we have $\delta(G_1, G_2) = 2p$.

Now we turn to trees. Note that trees can be characterized as connected graphs whose Hadwiger number is equal to 2.

We shall compare $\delta_C$ with $\delta_T$ defined above.

**Theorem 5.** Let $T_1, T_2$ be two trees with $n$ vertices. Then

$$\delta_C(T_1, T_2) \leq \delta_T(T_1, T_2).$$

**Proof.** Let $T$ be a tree, let $T_0$ be its subtree. By $V(T)$ (or $V(T_0)$) we denote the vertex set of $T$ (or $T_0$, respectively). For each vertex $x \in V(T_0)$ let $P(x)$ be the set consisting of $x$ and of all vertices $y \in V(T) - V(T_0)$ with the property that the path joining $x$ and $y$ in $T$ does not contain an inner vertex belonging to $V(T_0)$. The sets $P(x)$ for all $x \in V(T_0)$ form a partition $\mathcal{P}$ of $V(T)$ and each of them induces a connected subgraph of $T$. The contract of $T$ obtained by means of $\mathcal{P}$ is then isomorphic to $T_0$. We have proved that each subtree of a tree is also its contract. This implies the assertion. \(\square\)

**Theorem 6.** The maximum of the contraction distance between trees with $n \geq 3$ vertices is $n - 3$. The only trees having this distance are a star and a path.

**Proof.** Every tree with $n \geq 3$ vertices contains a subtree which is a path of length 2, i.e. with 3 vertices. Therefore $\delta_C(T_1, T_2) \leq \delta_T(T_1, T_2) \leq n - 3$ for any two trees $T_1, T_2$ with $n$ vertices. Every tree which is not a star contains a path of length 3 and every tree which is not a path contains a star with 3 edges as a subtree. Thus, if one of the trees $T_1, T_2$ is neither a path nor a star, then $\delta_C(T_1, T_2) \leq \delta_T(T_1, T_2) \leq n - 4$. \(\square\)

**Theorem 7.** Let $T_1, T_2$ be two trees with $n$ vertices. Then $\delta_C(T_1, T_2) \geq |d(T_1) - d(T_2)|$, where $d$ denotes the diameter.

**Proof.** Without loss of generality let $d(T_1) \geq d(T_2)$. In no contract of $T_2$ there may be a path of length greater than $d(T_2)$. Therefore in order to obtain a common contract of $T_1$ and $T_2$ we must contract at least $d(T_1) - d(T_2)$ edges, which implies the assertion. \(\square\)
Theorem 8. Let P be a path with n vertices, let T be a tree with n vertices. Then \( \delta_c(P, T) = n - 1 - d(T) \).

Proof. We have \( d(P) = n - 1 \) and thus \( \delta_c(P, T) \geq n - 1 - d(T) \) by Theorem 7. On the other hand, both P and T contain a subtree which is a path with \( d(T) + 1 \) vertices and thus \( \delta_c(P, T) \leq \delta(T, P, T) \leq n - 1 - d(T) \), which implies the assertion. \( \square \)

Theorem 9. Let \( t(T) \) denote the number of terminal vertices of T. Let \( T_1, T_2 \) be two trees with n vertices. Then \( \delta_c(T_1, T_2) \geq |t(T_1) - t(T_2)| \).

Proof. Without loss of generality let \( t(T_1) \geq t(T_2) \). Let \( T_{12} \) be a common contract of \( T_1 \) and \( T_2 \) and let \( \mathcal{P} \) be the partition of the vertex set \( V(T_2) \) of \( T_2 \) by means of which the tree \( T_{12} \) is obtained. If a class of \( \mathcal{P} \) contains only non-terminal vertices of \( T_2 \), then evidently this class is a non-terminal vertex of \( T_{12} \); hence \( t(T_{12}) \leq t(T_2) \). In order to obtain \( T_{12} \) from \( T_1 \) at least \( t(T_1) - t(T_2) \) terminal edges of \( T_1 \) must be contracted, which implies the assertion. \( \square \)

Theorem 10. Let S be a star with n vertices, let T be a tree with n vertices. Then \( \delta_c(S, T) = n - 1 - t(T) \).

Proof. We have \( t(S) = n - 1 \) and thus \( \delta_c(S, T) \geq n - 1 - t(T) \) by Theorem 9. On the other hand, a star with \( t(T) + 1 \) vertices is a common contract of \( S \) and \( T \). From \( S \) it is obtained by contracting arbitrary \( n - 1 - t(T) \) edges, from \( T \) by contracting all non-terminal edges. This implies the assertion. \( \square \)

In [8] there are theorems analogous to Theorem 8 and Theorem 10 for the edge rotation distance. Unfortunately, Theorem 6 from [8] is false. We will correct the paper [8] by proving the following theorem.

Theorem 11. Let P be a path with n vertices, let T be a tree with n vertices. Then \( \delta_R(P, T) = t(T) - 2 \).

Proof. When the transform \( P \) into \( T \) by edge rotations, at each edge rotation the degree of exactly one vertex may decrease. As \( P \) has two vertices of degree 1 and \( T \) has \( t(T) \) such vertices, we have \( \delta_R(P, T) \geq t(T) - 2 \). Now denote the terminal vertices of \( T \) by \( u_0, u_1, \ldots, u_{t(T) - 1} \). Let \( P_1 \) be the path in \( T \) connecting \( u_0 \) with \( u_1 \). For \( k \geq 2 \) let \( P_k \) be the path in \( T \) connecting \( u_k \) with the vertex \( v_k \) of the union of paths \( P_j \) for \( j = 1, \ldots, k - 1 \) whose distance from \( u_k \) in \( T \) is minimum. Let \( w_k \) be the vertex of \( P_k \) adjacent to \( v_k \) (\( w_k = u_k \) may hold). Let \( R_k \) for \( k = 1, \ldots, t(T) - 2 \) be the rotation of the edge \( w_{k+1}v_{k+1} \) to the position \( w_ku_{k+1} \). Evidently by the edge rotations \( R_1, \ldots, R_{t(T) - 2} \) the tree \( T \) is transformed into \( P \) and thus \( \delta_R(P, T) = t(T) - 2 \).
We see that the contraction distance of a tree $T$ from a path depends on the diameter of $T$ and its distance from a star depends on $t(T)$, while the edge rotation distance of $T$ from a path depends on $t(T)$ and its distance from a star depends on the maximum degree of $T$.

References


Souhrn

KONTRAKČNÍ VZDÁLENOST MEZI TŘÍDAMI ISOMORFISMU GRAFŮ

Bohdan Zelinka

Zavádí se jistá vzdálenost mezi třídami isomorfismu souvislých grafů; její definice je založena na maximálním počtu uzlů grafu, na který lze dané dva grafy převést kontrakcí.

Резюме

КОНТРАКЦИОННОЕ РАССТОЯНИЕ МЕЖДУ КЛАССАМИ ИЗОМОРФИЗМА ГРАФОВ

Bohdan Zelinka

Вводится расстояние между классами изоморфизма связных графов; его определение основано на максимальном числе вершин графа, на который можно два заданных графа стянуть.

Author's address: Šafářikova 280, 460 01 Liberec.