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ON DILATIONS AND CONTRACTIONS IN RIESZ GROUPS

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Summary. In the paper the notion of an \((m, n)\)-transposition in a partially ordered group is introduced (\(m\) and \(n\) are positive integers). If \(m < n\) (\(m > n\)), then an \((m, n)\)-transposition in an isolated partially ordered group is called a dilation (contraction). The main result establishes the relations between the \((m, n)\)-transpositions in an isolated abelian Riesz group \(G\) and the direct decompositions of \(G\). Further, it is shown that \((m, n)\)-transpositions in \(G\) preserve certain convex subsets of \(G\).

Keywords: \((m, n)\)-transposition, dilation, contraction, isometry, Riesz group.

AMS classification: 06F.

In [8] K. L. N. Swamy introduced the notion of an intrinsic metric in an abelian lattice ordered group \(H\) by putting \(d(x, y) = |x - y|\) for any \(x, y\) in \(H\). In [9], [10] K. L. N. Swamy studied isometries in an abelian lattice ordered group \(H\), i.e. bijections \(f: H \to H\) preserving the intrinsic metric of \(H\). Isometries in non-abelian lattice ordered groups have been studied by J. Jakubík [3], [4]. J. Jakubík proved that for every isometry \(f\) in a lattice ordered group \(H\) such that \(f(0) = 0\) there exists a uniquely determined direct decomposition \(H = A \times B\) of \(H\) such that \(f(x) = x(A) - x(B)\) is valid for each \(x \in H\) (\(x(A)\) and \(x(B)\) are the components of \(x\) in the direct factors \(A\) and \(B\), respectively). W. Ch. Holland [2] showed that the only intrinsic metrics in lattice ordered groups are the multiples \(n|x - y|\) of the metric \(|x - y|\). Isometries in Riesz spaces and \(f\)-rings have been studied by J. T. Pairó [11], [13]. In [5] J. Jakubík and M. Kolibiar extended the results on the relations between isometries and direct decompositions to abelian distributive multilattice groups. J. Rachůnek [7] generalized the notion of an intrinsic metric and an isometry to any partially ordered group and showed that every 2-isolated abelian Riesz group \(G\) is metrized by \(d(a, b) = |a - b|\) for each \(a, b \in G\) (where \(|x| = U(x, -x)\) for any \(x\) in \(G\)). Analogously (using the relation \(n|a| = |na|\)) it can be proved that in an isolated abelian Riesz group \(G\) the multiples \(n|x - y|\) of the metric \(|x - y|\) are intrinsic metrics in \(G\), too. In an \(f\)-ring \(A\) with a central superunity \(u\) (central subunity \(s\)) J. T. Pairó [12] studied the mappings \(F: A \to A\) satisfying \(|F(x) - F(y)| = u|x - y|\) \((|F(x) - F(y)| = s|x - y|\) for each \(x, y \in A\) and called them \(u\)-dilations (\(s\)-contractions) because \(|F(x) - F(y)| \geq |x - y|\) \((|F(x) - F(y)| \leq |x - y|\) holds for each \(x, y \in A\).
First we recall some notions and notation used in the paper. The set of all positive integers will be denoted by \( N \). Let \( H \) be a partially ordered group (notation po-group). The group operation will be written additively. We denote the set of all positive integers by \( N \).

Let \( H \) be a partially ordered group (notation po-group). The group operation will be written additively. We denote \( H^+ = \{ x \in H; x \geq 0 \} \).

If \( A \subseteq H \), we denote by \( U(A) \) and \( L(A) \) the set of all upper bounds and the set of all lower bounds of \( A \) in \( H \), respectively. For \( A = \{ a_1, \ldots, a_n \} \) we shall write \( U(a_1, \ldots, a_n) \) instead of \( U(\{ a_1, \ldots, a_n \}) \). For each \( a \in G \), \( |a| = U(a, -a) \). If \( A_1, \ldots, A_n \subseteq H \) then \( A_1 + \ldots + A_n = \{ a_1 + \ldots + a_n; \ a_1 \in A_1, \ldots, a_n \in A_n \} \). If \( A_1 = \ldots = A_n = A \), then we set \( nA = A_1 + \ldots + A_n \).

If \( m, n \in N \), then a bijection \( f: H \rightarrow H \) is called an \((m, n)\)-transposition in \( H \) if \( m|f(x) - f(y)| = n|x - y| \) for each \( x, y \in H \). A mapping \( f: H \rightarrow H \) is said to be a dilation (contraction) in \( H \) if \( |f(x) - f(y)| \leq |x - y| \) (\( |f(x) - f(y)| \geq |x - y| \)) for each \( x, y \in H \). If \( a \in H \), then the mapping \( f_a: H \rightarrow H \) defined by \( f_a(x) = x + a \) for each \( x \in H \) is called a right translation in \( H \). Every right translation in \( H \) is an isometry.

A mapping \( f: H \rightarrow H \) is called homogeneous if \( f(0) = 0 \).

We say that a po-group \( H \) is isolated if \( a \in H \) and \( m/ |(H) \) imply \( a \geq 0 \). A po-group \( H \) is called directed if \( U(x, y) = 0 \) and \( L(x, y) \neq 0 \) for each \( x, y \in H \). A Riesz group is any po-group \( H \) which is directed and has the Riesz interpolation property, i.e. for each \( a_i, b_j \in H \) \((i, j = 1, 2)\) such that \( a_i \leq b_j \) \((i, j = 1, 2)\) there exists \( c \in H \) such that \( a_i \leq c \leq b_j \) \((i, j = 1, 2)\). See [1].

1. **Lemma.** Let \( G \) be an isolated po-group. \( a, b \in G, m, n \in N \). Let \( m|a| = n|b|, m > n \). Then \( |b| \leq |a| \).

**Proof.** Let \( x \in |b| \). Then \( nx \in n|b| = m|a| \). Thus \( nx = y_1 + \ldots + y_m \), where \( y_1, \ldots, y_m \in |a| \). Since \( G \) is isolated, \( |a| \subseteq U(0) \). Then \( y_1 \geq 0 \) for \( i = 1, \ldots, m \). From the relations \( y_1 \geq a, y_1 \geq -a, \ldots, y_m \geq a, y_m \geq -a, y_{n+1} \geq 0, \ldots, y_m \geq 0 \) for the element \( nx = y_1 + \ldots + y_m \) we obtain \( nx \leq na, nx \geq -na \). Since \( G \) is isolated, we have \( x \in |a| \).

2. **Corollary.** Let \( G \) be an isolated po-group and let \( f \) be an \((m, n)\)-transposition in \( G \).

   (i) If \( m > n \), then \( f \) is a contraction.

   (ii) If \( m < n \), then \( f \) is a dilation.

If \( m > n \) \((m < n)\), then an \((m, n)\)-transposition in an isolated po-group is called an \((m, n)\)-contraction \((m, n)\)-dilation).

3. **Theorem.** Let \( f \) be an \((m, n)\)-transposition in a po-group \( H \). Then there exists a uniquely determined homogeneous \((m, n)\)-transposition \( h \) in \( H \) such that \( f(x) = h(x) + f(0) \) for each \( x \in H \).

**Proof.** If we put \( h(x) = f(x) - f(0) \) for each \( x \in H \), then \( h \) is clearly the required homogeneous \((m, n)\)-transposition.
So every \((m, n)\)-transposition can be uniquely represented as a composition of a homogeneous \((m, n)\)-transposition and a right translation.

4. **Theorem.** The set of all transpositions in a po-group \(H\) is a group with respect to the composition of mappings.

**Proof.** It is easy to verify that the composition of an \((m_1, n_1)\)-transposition and an \((m_2, n_2)\)-transposition is an \((m_2m_1, n_1n_2)\)-transposition. The inverse of an \((m, n)\)-transposition is an \((n, m)\)-transposition.

5. **Lemma.** Let \(H\) be a po-group, \(A, B_1, \ldots, B_n \subseteq H\) and let \(A = B_1 + \ldots + B_n\). An element \(u \in H\) is the least element of \(A\) if and only if \(u = u_1 + \ldots + u_n\), where \(u_i\) is the least element of \(B_i\) for \(i = 1, \ldots, n\).

**Proof.** a) Let \(u\) be the least element of \(A\) and let \(A = B_1 + \ldots + B_n\). Then \(u = u_1 + \ldots + u_n\), where \(u_i \in B_i\) for \(i = 1, \ldots, n\). Assume that \(u_i\) is not the least element of \(B_i\) for some \(i \in \{1, \ldots, n\}\). Then there exists \(u_i' \in B_i\) such that either \(u_i' \leq u_i\) or \(u_i' \parallel u_i\).

If \(u_i' \leq u_i\), then \(u_1 + \ldots + u_{i-1} + u_i' + u_{i+1} + \ldots + u_n \leq u_1 + \ldots + u_n = u\), which contradicts the assumption that \(u\) is the least element of \(A\).

If \(u_i' \parallel u_i\), then \(u_1 + \ldots + u_{i-1} + u_i' + u_{i+1} + \ldots + u_n \parallel u\), a contradiction.

Thus \(u_i\) is the least element of \(B_i\) for \(i = 1, \ldots, n\).

b) Let \(u_i\) be the least element of \(B_i\) for \(i = 1, \ldots, n\). Let \(v\) be an arbitrary element of \(A\). Then \(v = v_1 + \ldots + v_n\), where \(v_i \in B_i\) for \(i = 1, \ldots, n\). Since \(v_i \geq u_i\) for \(i = 1, \ldots, n\), we have \(v = v_1 + \ldots + v_n \geq u_1 + \ldots + u_n\). Thus \(u = u_1 + \ldots + u_n\) is the least element of \(A\).

6. **Theorem.** Let \(F\) be an isolated po-group, \(m, n \in \mathbb{N}\) and let \(f: F \to F\) be a mapping such that \(m|f(x) - f(y)| = n|x - y|\) for each \(x, y \in F\). Then \(f\) is an injection.

**Proof.** Let \(x, y \in F\) and let \(f(x) = f(y)\). Then \(n|x - y| = m|f(x) - f(y)| = m|0| = mU(0) = U(0)\). By 5, \(0 = nb\), where \(b\) is the least element of \(|x - y|\). Since \(F\) is isolated, we have \(b = 0\). Then the relations \(0 \geq x - y, 0 \geq y - x\) yield \(x = y\).

7. **Lemma.** Let \(f\) be a homogeneous \((m, n)\)-transposition in an isolated abelian directed group \(F\). Then

(i) for each \(c \in F\) there exists only one element \(d \in F\) such that \(mc = nd\),

(ii) for each \(c' \in F\) there exists only one element \(d' \in F\) such that \(nc' = md'\).

**Proof.** (i) Let \(b \in F^+\) and let \(a = f^{-1}(b)\). Then \(n|a| = m|f(a)| = m|b| = mU(b) = U(mb)\). Since \(mb\) is the least element of \(U(mb)\), 5 implies that \(mb = na_1\), where \(a_1\) is the least element of \(|a|\).
Let \( c \in F \). Since \( F \) is a directed group, \( c = c_1 - c_2 \) for some \( c_1, c_2 \in F^+ \) (cf. [1], Chap. II, Proposition 1). Then \( mc = mc_1 - mc_2 \). Further, there exist elements \( c'_1, c'_2 \in F \) such that \( mc_1 = nc'_1 \), \( mc_2 = nc'_2 \). Thus \( mc = n(c'_1 - c'_2) \).

Let \( mc = nd_1 \) and \( mc = nd_2 \) for some \( d_1, d_2 \in F \). Then \( n(d_1 - d_2) = 0 \). Since \( F \) is isolated, we have \( d_1 = d_2 \). (ii) Since the mapping \( f^{-1} \) is an \((n, m)\)-transposition, the assertion (ii) follows from (i).

Let \( G \) be a po-group, \( a \in G \). For \( m, n \in N \) let there exist only one element \( b \in G \) such that \( ma = nb \). Then \( b \) will be denoted by \( ma/n \).

If \( G \) is an isolated Riesz group, then the relation \( n|a| = |na| \) is valid for each \( a \in G \), \( n \in N \) (cf. [1], p. 114).

The following example shows that in a non-isolated abelian Riesz group \( G \) the following relations can be valid:

(i) \( m|a| \neq |ma| \) for some \( m \in N \), \( a \in G \),

(ii) \( n|b| = n|c| \) and \( |b| \neq |c| \) for some \( n \in N \), \( b, c \in G \).

Example. Let \( G_1 \) be the additive group of all real numbers with the natural order and let \( G_2 \) be the additive group of residue classes modulo 4 with the trivial order. Let \( G = G_1 \cdot G_2 \) be the lexicographic product of the po-groups \( G_1, G_2 \). Then \( G \) is a non-isolated abelian Riesz group.

Let \( a = (0, 1) \). Then \(-a = (0, 3), |a| = \{(x, y) \in G; x > 0\}, 2|a| = |a|, 2a = -2a = (0, 2), 2a = U((0, 2)) \). Since \( 2a = (0, 2) \in [2a], 2a \notin 2a, \) we have \( 2|a| \neq 2|a| \).

Let \( b = (0, 0), c = (0, 2) \). Then \( |b| = U((0, 0)), |c| = U((0, 2)), 2|b| = U((0, 0)), 2|c| = U((0, 0)) \). Thus \( 2|b| = 2|c|, \) but \( |b| \neq |c| \).

Throughout the rest of this paper let \( G \) be an isolated abelian Riesz group.

8. Lemma. Let \( a, b \in G \), \( n \in N \). If \( n|a| = n|b| \), then \( |a| = |b| \).

Proof. Let \( a, b \in G \), \( n \in N \) and let \( n|a| = n|b| \). If \( x \in |a| \), then \( nx \in n|a| = n|b| = |nb| \). From this we obtain \( nx \geq nb, nx \geq -nb \). Since \( G \) is isolated, we get \( x \geq b \).

Analogously, \( |b| \leq |a| \).

9. Lemma. Let \( f \) be a homogeneous \((m, n)\)-transposition in \( G \). For each \( x \in G \) define \( g(x) = m f(x)/n \). Then \( g \) is a homogeneous isometry in \( G \).

Proof. From 7 it follows that the mapping \( g \) is well defined. Let \( x, y \in G \) and let \( g(x) = g(y) \). Then \( m f(x)/n = m f(y)/n \). Thus \( m f(x) - f(y) = 0 \). Since \( G \) is isolated, we have \( f(x) = f(y) \). Hence \( x = y \). Let \( z \in G \). By 7, there exists \( nz/m \) in \( G \). Let \( u = f^{-1}(nz/m) \). Then \( g(u) = z \). Hence \( g \) is a bijection. Clearly \( g(0) = 0 \). Further we have \( n|a(x) - g(y)| = n|m f(x)/n - m f(y)/n| = n|n(m f(x)/n - m f(y)/n)| = = |m f(x) - f(y)| = m f(x) - f(y)| = n|x - y| \). By 8, we obtain \( |g(x) - g(y)| = |x - y| \).
The isometry defined in Lemma 8 is called the isometry associated with the given homogeneous \((m, n)\)-transposition.

If \(C = A \times B\) is a direct decomposition of a po-group \(C\), then for \(x \in C\) we denote by \(x(A)\) and \(x(B)\) the components of \(x\) in the direct factors \(A\) and \(B\), respectively.

10. **Theorem.** Let \(G\) be an isolated abelian Riesz group.

   (i) Let \(f\) be a homogeneous \((m, n)\)-transposition in \(G\). Then there exists a direct decomposition \(G = A \times B\) of \(G\) such that \(f(x) = nx(A)/m - nx(B)/m\) for each \(x \in G\).

   (ii) Let \(m, n \in \mathbb{N}\) and for each \(x \in G\) let the element \(nx|m\) in \(G\) exist. Let \(G = P \times Q\) be a direct decomposition of \(G\). If we put \(g(x) = nx(P)/m - nx(Q)/m\) for each \(x \in G\), then \(g\) is a homogeneous \((m, n)\)-transposition in \(G\).

**Proof.** (i) This is a consequence of 9 and Theorem 18 [6]. (ii) Clearly, \(g\) is a bijection and \(g(0) = 0\). It is easy to verify that \(|z| = |z(P)| + |z(Q)|\) for each \(z \in G\).

Let \(x, y \in G\). Then \(m|g(x) - g(y)| = m|nx(P)/m - nx(Q)/m + ny(Q)/m| = n|x(P) - x(Q)| - (y(P) - y(Q))| = n|x(P) - y(P)| + |-(x(Q) - y(Q))| = n|\langle x - y \rangle(P)| + |\langle x - y \rangle(Q)| = n|x - y|.

11. **Lemma.** Let \(f\) be a homogeneous isometry in \(G\), \(m, n \in \mathbb{N}\). For each \(x \in G\) let \(nx|m\) in \(G\) exist. If we put \(g(x) = nf(x)/m\) for each \(x \in G\), then \(g\) is a homogeneous \((m, n)\)-transposition in \(G\).

**Proof.** This is a consequence of Theorem 10.

12. **Theorem.** Let \(f\) be an \((m, n)\)-transposition in \(G\). Then \(f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))\) for each \(x, y \in G\).

**Proof.** If \(f\) is a translation, the assertion obviously holds. In view of 3 it suffices to consider the case when \(f\) is a homogeneous \((m, n)\)-transposition in \(G\).

Let \(g\) be the isometry associated with \(f\). Then \(g(z) = mf(z)/n\) for each \(z \in G\).

Let \(x, y \in G\). Let \(a \in U(L(x, y)) \cap L(U(x, y))\), \(u' \in L(f(x), f(y))\), \(v' \in U(f(x), f(y))\).

By 7, the elements \(u = mu'/n\), \(v = mv'/n\) in \(G\) exist. Since \(G\) is isolated, we have \(v \in U(g(x), g(y)), u \in L(g(x), g(y))\). By Theorem 22 [6], \(g(U(L(x, y)) \cap L(U(x, y))) = U(L(g(x), g(y))) \cap L(U(g(x), g(y)))\). Thus \(u \leq g(a) \leq v\). From this we obtain \(u' = nu|m \leq ng(a)/m = f(a) \leq nv|m = v'\). Therefore \(f(a) \in U(L(f(x), f(y))) \cap L(U(f(x), f(y)))\).

If we consider \(f^{-1}\) instead of \(f\), we can prove that \(U(L(f(x), f(y))) \cap L(U(f(x), f(y))) \subseteq f(U(L(x, y)) \cap L(U(x, y)))\).

13. **Theorem.** Let \(f\) be a homogeneous \((m, n)\)-transposition in \(G\) and let \(H \subseteq G\). Then \(H\) is a directed convex subset of \(G\) if and only if \(f(H)\) is a directed convex subset of \(G\).
Proof. Let $H$ be a directed convex subset of $G$. Let $f(x) \leq f(y) \leq f(z)$ for some $x, z \in H, y \in G$. By 7, the elements $m f(x)/n, m f(y)/n, m f(z)/n$ in $G$ exist. Let $g$ be the isometry associated with $f$. Since $G$ is isolated, we have $g(x) \leq g(y) \leq g(z)$. By Lemma 26 [6], $g(H)$ is a directed convex subset of $G$. Then $g(y) \in g(H)$. From this we get $y \in H$. Thus $f(y) \in f(H)$. Hence $f(H)$ is a convex subset of $G$.

Let $f(a), f(b) \in f(H)$. Then the elements $mf(a)/n = g(a), mf(b)/n = g(b)$ in $G$ exist. Since $g(H)$ is a directed subset of $G$, there exist elements $u, v \in H$ such that $g(u) \in L(g(a), g(b))$, $g(v) \in U(g(a), g(b))$. Since $G$ is isolated, we have $f(v) \in U(f(a), f(b))$, $f(u) \in L(f(a), f(b))$. Thus $f(H)$ is a directed subset of $G$.

If we consider $f^{-1}$, we can prove the sufficiency of the condition.

14. Lemma. Let $f$ be a homogeneous $(m, n)$-transposition in $G$ and let $g$ be the isometry associated with $f$. Let $C$ be a directed convex subgroup of $G$. Then $f(C) = g(C)$.

Proof. Let $C$ be a directed convex subgroup of $G$. By 10, $f$ and $g$ are group homomorphisms. From this and from 13 it follows that $f(C), g(C)$ are directed convex subgroups of $G$. Let $z \in g(C)$. Then there exist elements $u, v \in g(C)$ such that $v \in U(0, z)$, $u \in L(0, z)$. By 7, the elements $mv/n, mz/n, mu/n$ in $G$ exist. Since $G$ is isolated, we have $mu/n \leq mz/n \leq mv/n$. From the relations $0 \leq mv/n \leq mv$, $mu \leq mz/n \leq 0$ and from the convexity of $g(C)$ we obtain that $mv/n, mu/n \in g(C)$. Hence $mz/n \in g(C)$. Let $z' = g^{-1}(mz/n)$. Then $z' \in C, f(z') = g(z')/m = z \in f(C)$. Thus $g(C) \subseteq f(C)$.

Analogously we can prove that $g(C) \subseteq f(C)$.

15. Theorem. Let $f$ be a homogeneous $(m, n)$-transposition in $G$ and let $C$ be a directed convex subgroup of $G$. Then $f(C) = C$.

Proof. Let $g$ be the isometry associated with $f$. Let $x \in C$. Then there exist $u, v \in C$ such that $u \in L(x, 0), v \in U(x, 0)$. By 10, there exists a direct decomposition $G = A \times B$ of $G$ such that $g(z) = z(A) - z(B)$ for each $z \in G$. Then we have $v(A) \geq x(A), v(B) \geq x(B), v(A) \geq 0, v(B) \geq 0, u(A) \leq x(A), u(B) \leq x(B), u(A) \leq 0, u(B) \leq 0$. This implies $v \geq x(A) \geq u, v \geq x(B) \geq u$. By the convexity of $C, x(A), x(B) \subseteq C$. Since $x(A) - x(B) \subseteq C$ and $g(x(A) - x(B)) = x$, we have $C \subseteq g(C)$.

Let $y' \in g(C)$. Then $y' = g(y)$ for some $y \in C$. Since $y(A), y(B) \subseteq C$, we obtain $y' = y(A) - y(B) \in C$. Thus $g(C) \subseteq C$.

Therefore $g(C) = C$. In view of 14 we obtain $f(C) = C$.

16. Theorem. Let $f$ be an $(m, n)$-transposition in an isolated abelian po-group $F, a, c \in F, a \leq c$.

(i) If $f(a) \leq f(c)$, then $f([a, c]) = [f(a), f(c)]$.

(ii) If $f(a) \geq f(c)$, then $f([a, c]) = [f(c), f(a)]$. 

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Proof. (i) From the assumption we have \( c - a \geq 0, f(c) - f(a) \geq 0 \). Since 
\( n|a - c| = m|f(a) - f(c)| \) we have \( nU(c - a) = mU(f(c) - f(a)) \). Thus \( n(c - a) = 
m(f(c) - f(a)). \) Hence \( -mf(c) + nc = -mf(a) + na \).

Let \( b \in [a, c] \). Since \( b - a \geq 0 \), from \( n|b - a| = m|f(b) - f(a)| \) we get 
\( n(b - a) = d_1 + \ldots + d_m \), where \( d_1, \ldots, d_m \in |f(b) - f(a)| \). Then \( d_i \geq f(b) - f(a) \) for \( i = 1, \ldots, m \). Thus \( n(b - a) \geq m(f(b) - f(a)) \). This implies \( -mf(b) + nb \geq 
\geq -mf(a) + na = -mf(c) + nc \). Hence \( m(f(c) - f(b)) \geq n(c - b) \geq 0 \). Since \( F \) is isolated, we have \( f(c) \geq f(b) \). The relations \( c - b \geq 0, n|c - b| = m|f(c) - f(b)| \) imply that \( n(c - b) = m(f(c) - f(b)) \). Hence \( -mf(b) + nb = -mf(c) + nc = 
= -mf(a) + na \). Thus \( 0 \leq n(b - a) = m(f(b) - f(a)) \). Hence \( f(b) \geq f(a) \). Therefore 
\( f([a, c]) \subseteq [f(a), f(c)] \).

Let \( b' \in [f(a), f(c)] \), \( b = f^{-1}(b') \). Since \( f(b) - f(a) \geq 0 \), the relation \( n|b - a| = 
m|f(b) - f(a)| \) yields \( m(f(b) - f(a)) \geq nb - na \). Then \( -mf(b) + nb \leq 
\leq -mf(a) + na = -mf(c) + nc \). From this we get \( 0 \leq mf(c) - mf(b) \leq 
\leq nc - nb \). Since \( F \) is isolated, we have \( c \geq b \). Analogously we can prove that 
\( a \leq b \). Hence \( b \in [a, c] \). Therefore \([f(a), f(c)] \subseteq f([[a, c]]) \).

The assertion (ii) can be proved analogously.

17. Theorem. Let \( f \) be a homogeneous \((m, n)\)-transposition in \( G \), \( m > 1, n > 1 \).
Let \( g \) be the isometry associated with \( f \) and for each \( x \in G \) let \( x/n \) or \( x/m \) in \( G \) exist.
Then there exists a homogeneous \((1, n)\)-dilation \( f_1 \) and a homogeneous \((m, 1)\)-contraction \( f_2 \) such that 
\( f(x) = f_2(f_1(g(x))) \) for each \( x \in G \).

Proof. Let \( y \in G \). From 7 it follows that \( y/m \) exists in \( G \) if and only if \( y/n \) exists in \( G \). Put \( f_1(x) = nx \) and \( f_2(x) = x/m \) for each \( x \in G \). Since the identical mapping is a homogeneous isometry, 11 implies that \( f_1 \) is a homogeneous \((1, n)\)-dilation and \( f_2 \) is a homogeneous \((m, 1)\)-contraction in \( G \). Finally, we have 
\( f_2(f_1(g(x))) = 
= f_2(ny(x)) = ny(x)/m = f(x) \).

References

ON DILATIONS AND CONTRACTIONS IN RIESZ GROUPS

Sührn

V článku je zavedený pojem $(m, n)$-transpozície v čiastočne usporiadané gruppe $(m, n)$ sú kladné celé čísla). Pre $n > m (n < m)$ je $(m, n)$-transpozícia v izolovanej čiastočne usporiadané gruppe dilatáciou (kontrakciou).

Hlavný výsledok stanovuje vzťahy medzi $(m, n)$-transpozíciami v izolovanej abelovej Rieszovej gruppe $G$ a priamymi rozkladmi $G$. Ďalšie je ukázané, že $(m, n)$-transpozície v $G$ zachovávajú určité konvexné podmnožiny $G$.

Резюме

О ДИЛATAЦИЯХ И CЖАТИЯХ B ГРУППАХ РИССА

Sührn

В статье вводится понятие $(m, n)$-транспозиции в частично упорядоченной группе $(m$ и $n$— положительные целые числа). Если $n > m (m > n)$, то $(m, n)$-транспозиция в изолированной частично упорядоченной группе является дилатацией (сжатием).

Главный результат устанавливает соотношения между $(m, n)$-транспозициями в изолированной абелевой группе Рисса $G$ и прямыми разложениями $G$. Кроме того показано, что транспозиции в $G$ сохраняют некоторые выпуклые подмножества в $G$.

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