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HASSE'S OPERATOR AND DIRECTED GRAPHS

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In [1] the following problem by K. ČULÍK is given:

The graphs considered are sets together with a binary relation which is defined in them. If M is a set and $\sigma \subset M \times M$, then $T\sigma$ denotes the transitive closure of σ. Further we define $H\sigma = \{(u, v) \in \sigma$; there is no directed path $(w_1, \ldots, w_k)$ in $[M, \sigma]$ such that $k \geqq 3$ and $w_1 = u$, $w_k = v\}$. If $(w_1, \ldots, w_k)$ is a path in $[M, \sigma]$, then $(w_i, w_{i+1}) \in \sigma$ for $i = 1, 2, \ldots, k - 1$. We speak about the transitive closure operator T and Hasse's operator H. A partially ordered set is a graph $[M, \varrho]$, where $\varrho \subset \subset M \times M$ is an asymmetric and transitive relation (i.e. it is also irreflexive).

If M is a finite set, then $TH\varrho = \varrho$ and $[M, H\varrho]$ is said to be the Hasse's graph of the partially ordered set $[M, \varrho]$ (this is closely related to the well-known Hasse diagram of $[M, \varrho]$, see [2]). If M is an infinite set, the equality $TH\varrho = \varrho$ is not valid in general, but it always holds that $TH\varrho \subset \varrho$. Thus, if we put $x > y$ instead of $(x, y) \in \varrho$, we can define $[M, \varrho]$ as follows: $x_i \in M$ for $i = 0, 1, 2, \ldots$; $x_1 > x_2 > > \ldots > x_i > \ldots$ and $x_i > x_0$ for all $i = 1, 2, \ldots$ In this case $TH\varrho \neq \varrho$. On the other hand, if we add a new vertex w to M and define $u_i > w$ for all $i = 1, 2, \ldots$, but $w > u_0$, then for this new partially ordered set $[M', \varrho']$ we have $TH\varrho' = \varrho'$.

a) *Find necessary and sufficient conditions concerning ϱ for $TH\varrho = \varrho$, if $[M, \varrho]$ is an infinite partially ordered set. If $M = V^\infty$ and $\varrho = TC\mathfrak{R}(V^\infty$, C-operator and $\mathfrak{R}$ are defined in [3]), then ϱ is transitive, but need not be asymmetric.*

b) *Is it always true that $TC\mathfrak{R} = THTC\mathfrak{R}$? If not, what are necessary and sufficient conditions concerning $\mathfrak{R}$ that this equality holds?*

Remark. The vertices $w_1, \ldots, w_k$ need not be all different.

Here we shall give a solution of the problem a) and a partial solution of the problem b).

Before turning to the solution of the problem we shall define some concepts. If a partially ordered set $[M, \varrho]$ is given, then $N \subset M$ is a maximal chain of the set $[M, \varrho]$, if N is a chain (a totally ordered set) in the ordering induced by the ordering of the set M and there does not exist any subset of M which would contain N as a proper subset and would be a chain. If a, b are two elements of a partially ordered set $[M, \varrho]$ and $(a, b) \in \varrho$, then the closed interval $\langle a, b\rangle$ is by definition a set consisting of the elements a and b and all elements x for which simultaneously $(a, x) \in \varrho$ and $(x, b) \in \varrho$ holds.

From the above considerations it follows that we shall have to deal with directed graphs which do not contain multiple edges, but may contain loops.

Theorem 1. *Let $[M, \varrho]$ be an infinite partially ordered set. The equality $TH\varrho = \varrho$ holds if and only if for each two elements a, b of the set M such that $(a, b) \in \varrho$ there exists a finite maximal chain of the interval $\langle a, b\rangle$.*

Proof. Let the condition be fulfilled. Let a, b be arbitrary two elements of M for which $(a, b) \in \varrho$ holds. Therefore, there exists a finite maximal chain $N = \{a = x_1, x_2, \ldots, x_m = b\}$ of the interval $\langle a, b\rangle$ so that $(x_i, x_j) \in \varrho$ for $1 \leqq i < j \leqq m$. As N is a maximal chain of the interval $\langle a, b\rangle$, for no $i = 1, \ldots, m - 1$ there exists $y \in M$ such that $(x_i, y) \in \varrho$, $(y, x_{i+1}) \in \varrho$. In such a case $\{x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_m\}$ would be a chain which would be a subset of $\langle a, b\rangle$ and contain N as a proper subset. Thus, $(x_i, x_{i+1}) \in H\varrho$ for all $i = 1, \ldots, m - 1$. If we now apply the transitive closure operator, we get $(a, b) = (x_1, x_m) \in TH\varrho$. As we have chosen a and b quite arbitrarily, we have proved that $\varrho \subset TH\varrho$ and therefore $\varrho = TH\varrho$ (because we know that the inverse inclusion holds).

Now let $\varrho = TH\varrho$ hold. Let us have two elements a, b of M such that $(a, b) \in \varrho$; therefore also $(a, b) \in TH\varrho$. According to the definition of the transitive closure operator there exists a finite subset $N = \{x_1, \ldots, x_m\}$ of the set M such that $a = x_1$, $b = x_m$, $(x_i, x_{i+1}) \in H\varrho$ for $i = 1, \ldots, m - 1$. This set is a maximal chain of the interval $\langle a, b\rangle$. Actually, if a set N' existed which would contain N as a proper subset and would be a chain, then there would exist an element y such that $(x_i, y) \in \varrho$, $(y, x_{i+1}) \in \varrho$ for some i. Then there would exist a path consisting of the vertices $w_1 = x_i$, $w_2 = y$, $w_3 = x_{i+1}$ and thus $(x_i, x_{i+1}) \notin H\varrho$; in such a manner we obtain a contradiction.

We shall now generalize Theorem 1.

Theorem 2. *Let σ be a relation on the set M. The equality $THT\sigma = T\sigma$ holds if and only if the graph $[M, \sigma]$ is acyclic and for its transitive closure $[M, T\sigma]$ the condition of Theorem 1 holds.*

Proof. If $[M, \sigma]$ is acyclic, its transitive closure $[M, T\sigma]$ is a partially ordered set and we can apply Theorem 1. Thus, let us suppose that there exists at least one directed circuit D in $[M, \sigma]$; let its vertices be $a_1, \ldots, a_k$ and let $(a_i, a_{i+1}) \in \sigma$ for $i = 1, \ldots, k - 1$ and $(a_k, a_1) \in \sigma$ hold (Fig. 1). Then evidently for arbitrary i, j from the numbers $1, \ldots, k$ we have $(a_i, a_j) \in T\sigma$, because a directed path from a_i to a_j exists which is a subgraph of the circuit D. The subgraph of the graph $[M, T\sigma]$ generated by the vertices $a_1, \ldots, a_k$ is therefore a complete directed graph. Further, for arbitrary i, j from the numbers $1, \ldots, k$ we have $(a_i, a_j) \notin HT\sigma$; for arbitrary l from the numbers $1, \ldots, k$ particularly $(a_i, a_l) \in T\sigma$, $(a_l, a_k) \in T\sigma$, i.e. there exists a directed path with the vertices $w_1 = a_i$, $w_2 = a_l$, $w_3 = a_j$. The subgraph of the graph $[M, HT\sigma]$ generated by the vertices $a_1, \ldots, a_k$ is therefore a graph without edges. If $(a_i, a_j) \in THT\sigma$ held for some i, j from the numbers $1, \ldots, k$, this would

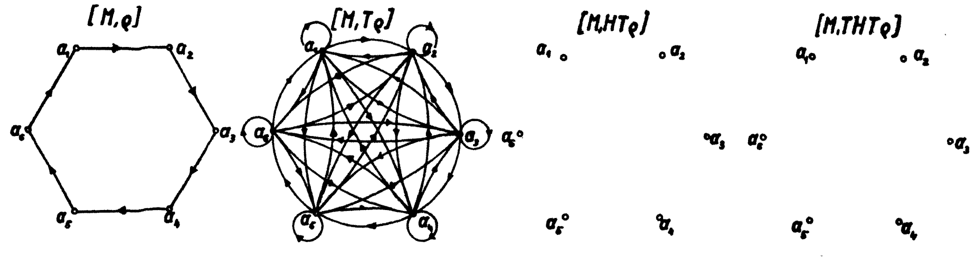

Fig. 1.

mean that there exist elements $b_1, \ldots, b_m$ of M such that $(a_i, b_1) \in HT\sigma$, $(b_m, a_j) \in$ $\in HT\sigma$ and $(b_n, b_{n+1}) \in HT\sigma$ for $n = 1, \ldots, m - 1$. Let p be the least positive integer such that the element b_p is equal to some of the elements $a_1, \ldots, a_k$. Thus, $b_p = a_q$ for some q, $1 \leqq q \leqq k$, and none of the elements $b_1, \ldots, b_{p-1}$ is equal to any of the elements $a_1, \ldots, a_k$. Without loss of generality let $q > i$. The elements $a_1, \ldots, a_i$, $b_1, \ldots, b_{p-1}, a_q, \ldots, a_k$ therefore form a directed circuit in $[M, \sigma]$ (as $HT\sigma \subset \sigma$), so that the subgraph of the graph $[M, HT\sigma]$ generated by them will be without edges, which leads to a contradiction. Consequently, also the subgraph of the graph $[M, THT\sigma]$ generated by the vertices $a_1, \ldots, a_k$ is without edges. That is why $THT\sigma \neq T\sigma$.

About the graph $[V^\infty, C\mathfrak{R}]$ we shall give only a few remarks. At first we shall give definitions. V is a finite set called the alphabet, V^∞ is the set of all words on this alphabet. $\mathfrak{R}$ is a certain finite relation on V^∞ and its elements are called rules. $C\mathfrak{R}$ is a relation consisting of all pairs (xay, xby), where $(a, b) \in \mathfrak{R}$ and x, y are arbitrary words from V^∞ (they may be empty).

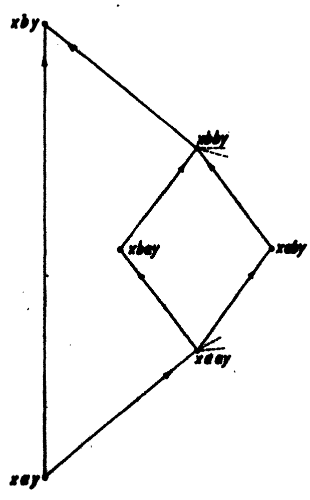

Fig. 2.

The necessary condition for $THTC\mathfrak{R} = TC\mathfrak{R}$ is that $[V^\infty, C\mathfrak{R}]$ is acyclic. We can prove that this condition is not sufficient. Let us have $V = \{a, b\}$, $\mathfrak{R} = \{(a, aa), (a, b), (bb, b)\}$. Then $(a, b) \in TC\mathfrak{R}$ but $(a, b) \notin HTC\mathfrak{R}$, because the directed path with the vertices $w_1 = a$, $w_2 = aa$, $w_3 = ab$, $w_4 = bb$, $w_5 = b$ exists. However, at every inference of b from a we must apply the rule $(a, b) \in \mathfrak{R}$ as other two rules would not suffice. If we have an arbitrary directed path with the vertices $a = c_1, \ldots$

..., $c_k = b$, where $(c_i, c_{i+1}) \in C\mathfrak{R}$ for $i = 1, \ldots, k - 1$, we have $c_i = xay$, $c_{i+1} = xby$ for some i; therefore, $(c_i, c_{i+1}) \notin HTC\mathfrak{R}$, as also $(a, b) \notin HTC\mathfrak{R}$. Thus, there does not exist a path $a = d_1, \ldots, d_l = b$ such that we had $(d_i, d_{i+1}) \in HTC\mathfrak{R}$ for each $i = 1, \ldots, l - 1$ (Fig. 2).

An open problem remains, what is the necessary and sufficient condition for $\mathfrak{R}$ under which the graph $[V^\infty, C\mathfrak{R}]$ might be acyclic and the graph $[V^\infty, TC\mathfrak{R}]$ might fulfill the condition of Theorem 1.

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[2] *G. Birkhoff:* Lattice Theory. New York 1948.

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Výtah

HASSEŮV OPERÁTOR A ORIENTOVANÉ GRAFY

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V článku se zkoumá orientovaný graf $[M, \sigma]$ jako množina M s binární relací σ. Uvažují se dva operátory, operátor transitivního uzávěru T a Hasseův operátor H, který je definován takto: platí $H\sigma = \{(u, v) \in \sigma$; neexistuje orientovaný tah $(w_1, \ldots, w_k)$ v $[M, \sigma]$ takový, že $k \geqq 3$ a $w_1 = u$, $w_k = v\}$. Dokazují se dvě věty, které jsou částečným řešením problému K. Čulíka.

Věta 1. *Budiž $[M, \sigma]$ nekonečná částečně uspořádaná množina. Platí $TH\varrho = \varrho$ právě tehdy, existuje-li ke každým dvěma prvkům a, b množiny M, pro něž $(a, b) \in \varrho$, konečný maximální řetězec, který je podmnožinou intervalu $\langle a, b \rangle$.*

Věta 2. *Budiž σ relace na množině M. Rovnost $THT\sigma = T\sigma$ platí právě tehdy, jestliže graf $[M, \sigma]$ je acyklický a pro jeho transitivní uzávěr $[M, T\sigma]$ platí podmínka z věty* 1.

Závěrem se výsledky aplikují na matematickou lingvistiku.

Резюме

ОПЕРАТОР ХАССЕ И НАПРАВЛЕННЫЕ ГРАФЫ

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В статье исследуется направленный граф $[M, \sigma]$ как множество M с бинарным отношением σ. Рассматриваются два оператора — оператор транзитивного замыкания T и оператор Хассе H, который определен следующим способом: справедливо $H\sigma = \{(u, v) \in \sigma$; не существует направленного пути $(w_1, \ldots, w_k)$ в $[M, \sigma]$ такого, что $k \geqq 3$ и $w_1 = u, w_k = v\}$. Доказываются две теоремы, которые служат частичным решением проблемы К. Чулика.

Теорема 1. *Пусть $[M, \varrho]$ — бесконечное частично упорядоченное множество. Справедливо $TH\varrho = \varrho$ тогда и только тогда, если для всяких двух элементов a, b множества M, для которых $(a, b) \in \varrho$, существует конечная максимальная цепь, которая является подмножеством интервала $\langle a, b \rangle$.*

Теорема 2. *Пусть σ — отношение на множестве M. Равенство $THT\sigma = T\sigma$ имеет место тогда и только тогда, когда граф $[M, \sigma]$ ациклический и для его транзитивного замыкания $[M, T\sigma]$ выполнено условие из теоремы* 1.

В конце статьи применяются результаты к математической лингвистике.