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Hereditary radical classes of linearly ordered groups

Časopis pro pěstování matematiky, Vol. 108 (1983), No. 2, 199--207

Persistent URL: http://dml.cz/dmlcz/108408

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The study of radical classes and semisimple classes of linearly ordered groups was begun by Chehata and Wiegandt [1]. The basic properties of the lattice $\mathcal{R}$ of all radical classes of linearly ordered groups were described in [3]; for analogous questions concerning semisimple classes cf. [4]. In the papers [5], [7] and [8] radical classes and semisimple classes of abelian linearly ordered groups were dealt with.

In [3] and [4] it was proved that the lattice $\mathcal{R}$ has no atoms, no antiatoms and fails to be modular.

A radical class $X \in \mathcal{R}$ is said to be hereditary if, whenever $G \in X$ and $H$ is a convex subgroup of $G$, then $H \in X$. The collection of all hereditary radical classes will be denoted by $\mathcal{R}_h$.

In this note it will be shown that $\mathcal{R}_h$ (partially ordered by inclusion) is a complete distributive lattice. In fact, $\mathcal{R}_h$ fulfils the infinite distributive law

$$A \land (\lor B) = \lor (A \land B),$$

hence $\mathcal{R}_h$ is a Brouwer lattice. The corresponding dual infinite distributive law does not hold in $\mathcal{R}_h$. Further, it will be proved that $\mathcal{R}_h$ has infinitely many atoms and that the collection $\mathcal{P}$ of all prime intervals of the lattice $\mathcal{R}_h$ is a proper collection. Thus some properties of the lattice $\mathcal{R}_h$ are analogous to those of the lattice of all radical classes of $l$-groups [2] or the lattice of all torsion classes of $l$-groups (cf. Martinez [6]).

The collection of all principal elements of $\mathcal{R}_h$ will be denoted by $\mathcal{R}_{hp}$. It will be shown that if $X \in \mathcal{R}_h$, $Y \in \mathcal{R}_{hp}$ and $X \leq Y$, then $X \in \mathcal{R}_{hp}$. If $I \neq \emptyset$ is a set and $\{X_i\}_{i \in I} \subseteq \mathcal{R}_{hp}$, then $\bigvee_{i \in I} X_i$ belongs to $\mathcal{R}_{hp}$ as well. (Let us remark that analogous results do not hold for principal elements of the lattice of all radical classes of abelian linearly ordered groups; cf. [5].)

1. BASIC NOTIONS

A collection $X$ will be said to be propre if there exists a one-to-one mapping of the class of all cardinals into $X$. 

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The group operation in a linearly ordered group will be denoted by +; the com-
mutativity of this operation is not assumed. We recall some definitions; cf. [1].

Let \( \mathcal{G} \) be the class of all linearly ordered groups. When considering a subclass \( X \) of \( \mathcal{G} \) we always suppose that \( X \) is closed with respect to isomorphisms and that the zero linearly ordered group \( \{0\} \) belongs to \( X \).

A subclass \( X \) of \( \mathcal{G} \) is said to be closed with respect to transfinite extensions if, whenever \( G \in \mathcal{G} \) and

\[
\{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \delta)
\]

is an ascending chain of convex normal subgroups of \( G \) such that

\[
G_\beta/\bigcup_{\gamma < \beta} G_\gamma \in X \quad \text{for each} \quad \beta < \delta,
\]

then \( \bigcup_{\alpha < \delta} G_\alpha \) belongs to \( X \).

We also say that the linearly ordered group \( \bigcup_{\alpha < \delta} G_\alpha \) is a transfinite extension of linearly ordered groups \( G_\beta'(b < \delta) \), where \( G_\beta' \) is isomorphic to \( G_\beta/\bigcup_{\gamma < \beta} G_\gamma \) for each \( \beta < \delta \).

1.1. Definition. A class \( X \) of linearly ordered groups is called a radical class, if

(a) \( X \) is closed under homomorphisms, and

(b) \( X \) is closed with respect to transfinite extensions.

We denote by \( \mathcal{R} \) the collection of all radical classes. Further, let \( \mathcal{R}_h \) be the collection of all hereditary radical classes. Both \( \mathcal{R} \) and \( \mathcal{R}_h \) are partially ordered by inclusion. Then \( \mathcal{G} \) is the greatest element in both \( \mathcal{R} \) and \( \mathcal{R}_h \); the trivial variety \( R_0 \) containing all one-element \( l \)-groups is the least element in both \( \mathcal{R} \) and \( \mathcal{R}_h \).

If \( \{A_i\}_{i \in I} \) is a non-empty collection of hereditary radical classes, then \( \bigcap_{i \in I} A_i \) also is a hereditary radical class. Thus \( \mathcal{R}_h \) is a complete lattice. The lattice operations in \( \mathcal{R}_h \) will be denoted by \( \wedge \) and \( \vee \). The operation \( \wedge \) in \( \mathcal{R}_h \) coincides with the intersection of classes.

Let \( Y \subseteq \mathcal{G} \) and \( G \in \mathcal{G} \). The intersection of all hereditary radical classes \( X \) with \( Y \subseteq X \) will be denoted by \( T_h(X) \). Similarly, the intersection of all hereditary radical classes \( Z \) with \( G \in Z \) is denoted by \( T_h(G) \); the hereditary radical class \( T_h(G) \) is said to be principal. We denote by \( \mathcal{R}_{hp} \) the collection of all principal hereditary radical classes.

2. THE OPERATION \( \vee \) IN THE LATTICE \( \mathcal{R}_h \)

Let \( X \) be a subclass of \( \mathcal{G} \). We denote by

- \( \text{Hom} X \) — the class of all homomorphic images of linearly ordered groups belonging to \( X \);
- \( \text{Sub} X \) — the class of all convex subgroups of linearly ordered groups belonging to \( X \);
Ext \( X \) — the class of all transfinite extensions of linearly ordered groups belonging to \( X \).

Now we define for each ordinal \( \alpha \) the class \( \text{Ext}_\alpha X \) by induction as follows. We put \( \text{Ext}_1 X = \text{Ext} X \); if \( \alpha > 1 \), then we set

\[
\text{Ext}_\alpha X = \text{Ext} \bigcup_{\beta < \alpha} \text{Ext}_\beta X.
\]

Next we denote

\[
\text{ext } X = \bigcup_{\alpha} \text{Ext}_\alpha X,
\]

where \( \alpha \) runs over the class of all ordinals.

**2.1. Theorem.** Let \( X \) be a subclass of \( \mathcal{G} \). Then \( T_\alpha(X) = \text{ext} \text{Hom Sub} X \).

**Proof.** Denote \( \text{ext} \text{Hom Sub} X = Z \). Clearly \( Z \subseteq T_\alpha(X) \) and \( X \subseteq Z \). Hence it suffices to prove that \( Z \) is a hereditary radical class. Thus we have to verify that \( Z \) fulfills the following conditions: (i) \( \text{Ext} Z \subseteq Z \), (ii) \( \text{Sub} Z \subseteq Z \); (iii) \( \text{Hom} Z \subseteq Z \).

For each subclass \( Z_1 \) of \( \mathcal{G} \) we have \( \text{Ext} \text{ext} Z_1 = \text{ext} Z_1 \), hence (i) is valid. In [3] (Lemma 2.1) it was proved that for each subclass \( Z_2 \) of \( \mathcal{G} \) the relation

\[
\text{Hom} \text{ext} \text{Hom} Z_2 = \text{ext} \text{Hom} Z_2
\]

holds; therefore (iii) holds as well.

Let \( G \in \mathcal{Z} \) and let \( H \) be a convex subgroup of \( G \) with \( H \subseteq G \). Hence there is an ordinal \( \alpha \) such that \( G \in \text{Ext}_\alpha \text{Hom Sub} X \). Thus it suffices to verify that for each ordinal \( \alpha \) we have

\[
(1) \quad \text{Sub} \text{Ext}_\alpha \text{Hom Sub} X \subseteq \text{Ext}_\alpha \text{Hom Sub} X.
\]

a) Let \( \alpha = 1 \). There is an ascending chain of convex normal subgroups

\[
(2) \quad \{0\} = G_1 \subseteq G_2 \subseteq \ldots \subseteq G_\alpha \subseteq \ldots \quad (\alpha < \delta)
\]

of \( G \) such that

\[
(3) \quad \bigcup_{\beta < \delta} G_\beta = G
\]

and for each \( \beta < \delta \), \( G_\beta / \bigcup_{\gamma < \beta} G_\gamma \in \text{Hom Sub} X \). Let \( \lambda \) be the first ordinal with \( \lambda < \delta \) and \( G_\lambda \supseteq H \). Denote \( H_\alpha = H \cap G_\alpha \) for each \( \alpha < \delta \). Then \( \{H_\alpha\} \) \((\alpha < \delta)\) is an ascending chain of convex normal subgroups of \( H \) and \( \bigcup_{\alpha < \delta} H_\alpha = H \). If \( \beta < \lambda \), then

\[
G_\beta / \bigcup_{\gamma < \beta} G_\gamma = H_\beta / \bigcup_{\gamma < \beta} H_\gamma;
\]

if \( \beta > \lambda \), then \( H_\beta / \bigcup_{\gamma < \beta} H_\gamma = \{0\} \). In the case \( \beta = \lambda \) we have

\[
H_\beta / \bigcup_{\gamma < \beta} H_\gamma \in \text{Sub} \{G_\beta / \bigcup_{\gamma < \beta} G_\gamma\} \subseteq \text{Sub} \text{Hom Sub} X = \text{Hom Sub} X,
\]

thus for \( \alpha = 1 \) the relation (1) holds. (We use the well-known relation \( \text{Sub Hom} Y \subseteq \subseteq \text{Hom Sub} Y \) which is valid for each \( Y \subseteq \mathcal{G} \).)
b) Assume that \( x > 1 \) and that (1) holds for each ordinal less than \( x \). Then there is an ascending chain of convex normal subgroups (2) of \( G \) such that (3) is valid and for each \( \beta < \delta \) there is an ordinal \( \gamma(\beta) < x \) having the property

\[
G_{\beta}/\bigcup_{\gamma < \beta} G_{\gamma} \in \text{Ext}_{\gamma(\beta)} \text{Hom Sub } X.
\]

Let \( \lambda \) and \( H_\lambda \) \((\alpha < \gamma)\) be as in part a). The cases \( b < \lambda \) and \( b > \lambda \) are analogous as in a). Let \( b = \lambda \). Then

\[
H_{\beta}/\bigcup_{\gamma < \beta} H_{\gamma} \in \text{Sub } \{G_{\beta}/\bigcup_{\gamma < \beta} G_{\gamma}\} \subseteq \text{Sub Ext}_{\gamma(\beta)} \text{Hom Sub } X = \text{Ext}_{\gamma(\beta)} \text{Hom Sub } X,
\]

hence (1) is valid for each ordinal \( x \), which completes the proof.

2.2. Theorem. Let \( I \) be a nonempty class and for each \( i \in I \) let \( X_i \) be a hereditary radical class. Then \( \bigvee_{i \in I} X_i = \text{ext } \bigcup_{i \in I} X_i \).

Proof. From 2.1 it follows immediately that the relation

\[
\bigvee_{i \in I} X_i = \text{ext } \bigcup_{i \in I} X_i
\]

is valid. Since \( X_i \) are hereditary radical classes, we have \( \text{Hom Sub } X_i = X_i \), therefore

\[
\bigvee_{i \in I} X_i = \text{ext } \bigcup_{i \in I} X_i.
\]

From 2.2 and [3] (Thm. 2.3) we obtain:

2.2.1. Corollary. \( R_h \) is a closed sublattice of the complete lattice \( \mathcal{R} \).

2.3. Theorem. Let \( A \in \mathcal{R}_h \), \( \{B_i\}_{i \in I} \subseteq \mathcal{R}_h \). Then

\[
A \land (\bigvee_{i \in I} B_i) = \bigvee_{i \in I} (A \land B_i).
\]

Proof. It suffices to verify that \( A \land (\bigvee_{i \in I} B_i) \leq \bigvee_{i \in I} (A \land B_i) \). Let \( G \in A \land (\bigvee_{i \in I} B_i) \). Hence \( G \in A \) and \( G \in \bigvee_{i \in I} B_i \). In view of 2.2, \( G \in \text{ext } \bigcup_{i \in I} B_i \). Thus \( G \) is constructed by the operation \( \text{ext} \) from certain linearly ordered groups \( G_{ij} \) \((i \in I, j \in K_i)\) such that \( G_i \) belongs to \( B_i \) for each \( i \in I \) and each \( j \in K_i \).

According to the definition of \( \text{ext} \), for each \( G_{ij} \) there exists a normal convex subgroup \( H_{ij} \) of \( G \) and a homomorphic image \( G'_{ij} \) of \( H_{ij} \) such that \( G'_{ij} \) is isomorphic to \( G_{ij} \). Because \( A \) is hereditary the linearly ordered group \( H_{ij} \) belongs to \( A \) and hence \( G_{ij} \in A \). Thus \( G_{ij} \in A \land B_i \) for each \( i \in I \) and each \( j \in K_i \). Therefore \( G \in \text{ext } \bigcup_{i \in I} \). \( \land (A \land B_i) = \bigvee_{i \in I} (A \land B_i) \).

The following example shows that the relation

\[
A \lor (\bigwedge_{i \in I} B_i) = \bigwedge_{i \in I} (A \lor B_i)
\]

does not hold in general in the lattice \( \mathcal{R}_h \). (The symbols \( \Gamma_{ja} \) \( G_j \) and \( G_1 \circ G_2 \) denote lexicographic products of linearly ordered groups; cf., e.g., [5].)
2.4. Example. Let $N$ be the set of all positive integers with the natural linear order. Let $J$ be the linearly ordered set dual to $N$ and for each $j \in J$ let $G_j$ be an archimedean linearly ordered group, $G_j \neq \{0\}$, such that $G_{j(1)}$ and $G_{j(2)}$ fail to be isomorphic whenever $j(1)$ and $j(2)$ are distinct elements of $J$. For each $j \in J$ let $J_j = \{k \in J : k \leq j\}$ (with the induced linear order). Put

\[
G = \Gamma_{j \in J} G_j, \\
G_{(j)} = \Gamma_{k \in J} G_k \quad \text{for each} \quad j \in J, \\
A = \bigvee_{j \in J} \mathcal{T}(G_j), \\
B_j = \mathcal{T}(G_{(j)}) \quad \text{for each} \quad j \in J.
\]

Then we have $G \notin A$, $\bigwedge_{j \in J} B_j = R_0$, hence

\[
A \lor (\bigwedge_{j \in J} B_j) = A
\]

and thus $G \notin A \lor (\bigwedge_{j \in J} B_j)$.

On the other hand, $G \in A \lor B_j$ for each $j \in J$, hence

\[
G \in \bigwedge_{j \in J} (A \lor B_j)
\]

and therefore $A \lor (\bigwedge_{j \in J} B_j) = \bigwedge_{j \in J} (A \lor B_j)$.

2.5. Lemma. Let $X \subseteq \mathcal{G}$, $H \in \mathcal{T}(X)$, $H \neq \{0\}$. Then there exists a convex subgroup $H_1$ of $H$ with $H_1 \neq \{0\}$ such that $H_1 \in \text{Hom Sub } X$.

Proof. In view of 2.1 we have $H \in \text{ext Hom Sub } X$, hence there is an ordinal $\tau$ such that $H \in \text{Ext}_\tau \text{Hom Sub } X$. Thus there is an ordinal $\tau < \tau$ having the property that there exists a convex subgroup $H'$ of $H$ with $H' \neq \{0\}$ such that $H' \in \text{Ext}_\tau \text{Hom Sub } X$.

Now let $\chi$ be the first ordinal having the property that there is a convex subgroup $H''$ of $H$ with $H'' \neq \{0\}$ such that $H'' \in \text{Ext}_\chi \text{Hom Sub } X$. Assume that $\chi > 1$. Then there is $\chi' < \chi$ such that there exists a convex subgroup $H^* \neq \{0\}$ of $H''$ with $H^* \in \text{Ext}_{\chi'} \text{Hom Sub } X$. Since $H^*$ is a convex subgroup of $H$, we have arrived at a contradiction. Hence $\chi = 1$. Therefore there is a convex subgroup $H_1 \neq \{0\}$ of $H''$ such that $H_1 \in \text{Hom Sub } X$, which completes the proof.

3. ATOMS IN $\mathcal{R}_h$

3.1. Proposition. Let $G \in \mathcal{G}$, $G \neq \{0\}$. Assume that $G$ is archimedean. Then $T_h(G)$ is an atom in the lattice $\mathcal{R}_h$.

Proof. We have $R_0 < T_h(G)$. Let $A \in \mathcal{R}_h$, $R_0 < A \leq T_h(G)$. There exists $H \in A$ with $H \neq \{0\}$. In view of 2.1 we have $T_h(G) = \text{ext Hom Sub } \{G\}$. Since $G$ is archimedean, $\text{Hom Sub } \{G\}$ is the class of all linearly ordered groups $G'$ such that either $G' = \{0\}$ or $G'$ is isomorphic to $G$. Hence $H$ can be constructed by the operation $\text{ext}$.
from a system of linearly ordered groups \( G_i \) \((i \in I)\) such that each \( G_i \) is isomorphic to \( G \). Let \( i \in I \) be fixed. There exists a normal convex subgroup \( H_i \) of \( G \) and a homomorphic image \( G'_i \) of \( H_i \) such that \( G'_i \) is isomorphic to \( G \). Since \( A \) is hereditary, we have \( H_i \in A \) and thus \( G'_i \in A \). Therefore \( G \in A \) and hence \( A = T_h(G) \).

Because there is an infinite set of mutually nonisomorphic archimedean linearly ordered groups, 3.1 implies:

**3.2. Corollary.** The class of all atoms of the lattice \( R_h \) is infinite.

**3.3. Proposition.** Let \( X \in R_h \), \( X \neq R_0 \). Then there exists an archimedean linearly ordered group \( H \neq \{0\} \) such that \( T_h(H) \leq X \).

**Proof.** There exists \( G \in X \) such that \( G \neq \{0\} \). Choose \( g \in G \), \( g > 0 \) and let \( H' = \{H_i\}_{i \in I} \) be the set of all convex subgroups of \( G \) not containing the element \( g \). Let \( H'_1 \) be the convex subgroup of \( G \) generated by \( g \). Because the set \( H' \) is linear ordered, \( H' \) has a unique maximal element \( H_2 \). Then \( H_2 \) is the largest proper convex subgroup of \( H_1 \). Hence \( H_2 \) is a normal subgroup in \( H_1 \). Therefore \( H = H_1/H_2 \) is \( o \)-simple and thus it is archimedean. Clearly \( H \neq \{0\} \). Now we have \( T_h(H) = T_h(H_1/H_2) \leq T_h(G) \leq T_h(X) \).

From 3.1 and 3.3 we infer:

**3.4. Theorem.** Let \( X \in R_h \). Then the following conditions are equivalent:

(i) \( X \) covers \( R_0 \) in the lattice \( R_h \).
(ii) There is an archimedean linearly ordered group \( H \neq \{0\} \) such that \( X = T_h(G) \).

Let \( A_0 \) be a set of non-zero archimedean linearly ordered groups such that (a) if \( G_1 \) and \( G_2 \) are distinct elements of \( A_0 \), then \( G_1 \) is not isomorphic to \( G_2 \), and (b) for each non-zero archimedean linearly ordered group \( G \) there is \( G' \) in \( A_0 \) such that \( G \) is isomorphic to \( G' \). Put

\[
X_0 = \bigvee_{G \in A_0} T_h(G).
\]

A collection \( X \) will be said to be small if there exists a set \( Y \) and a mapping of \( Y \) onto \( X \).

**3.5. Proposition.** Let \( \mathcal{G}_1 = [R_0, X_0] \) (the interval taken in \( R_h \)). Then

(i) \( \mathcal{G}_1 \) is a small collection;
(ii) \( \mathcal{G}_1 \) is a complete atomic Boolean algebra; the collection of atoms of \( \mathcal{G}_1 \) is \( \{T_h(G)\}_{G \in A_0} \).

**Proof.** \( \mathcal{G}_1 \) is obviously a complete lattice and in view of 2.3, \( \mathcal{G}_1 \) is distributive. From 3.4 it follows that \( A'_0 = \{T_h(G)\}_{G \in A_0} \) is the collection of all atoms of \( \mathcal{G}_1 \). Let \( R_0 \neq X \in \mathcal{G}_1 \) and let \( X' = \{T_h(G) : G \in A_0 \cap X\} \). Then

\[
X = X \land X_0 = X \land (\bigvee_{G \in A_0} T_h(G)) = \bigvee_{G \in A_0 \cap X} (X \land T_h(G)) = \bigvee_{G \in A_0 \cap X} (X \land T_h(G)) = \sup X'.
\]
Moreover, if \( X'' \subseteq A_0 \) and \( \text{sup} \ X'' = X \), then 2.3 implies that \( X' = X'' \). Hence \( \mathcal{G}_1 \) is isomorphic to the Boolean algebra of all subsets of the set \( A_0 \).

3.6. Lemma. Let \( X \in \mathcal{G}_1, \ X \neq R_0 \). Let \( I \) be a linearly ordered set isomorphic to the set of all negative integers (with the natural order). Let \( G = \Gamma_{\text{let}} G_i \), where each \( G_i \) belongs to \( A_0 \cap X \). Assume that for each \( G' \in A_0 \cap X \) and each \( j \in I \) there is \( i \in I \) with \( i < j \) such that \( G' \) is isomorphic to \( G_i \). Then

(i) \( T_\mathcal{G}(G) \) covers \( X \),
(ii) \( T_\mathcal{G}(G) \) does not belong to \( \mathcal{G}_1 \),
(iii) \( T_\mathcal{G}(G) \land T_\mathcal{G}(G') = R_0 \) whenever \( G' \in A_0 \) and \( G' \notin X \).

Proof. We apply the same notations as in the proof of 3.5. For each \( G' \in A_0 \cap X \) we have \( T_\mathcal{G}(G') \leq T_\mathcal{G}(G) \), hence \( X = \bigvee_{G' \in A_0 \cap X} \mathcal{G}_t(G') \leq T_\mathcal{G}(G) \). In view of 2.5, \( T_\mathcal{G}(G) \) does not belong to \( \mathcal{G}_1 \) and thus \( X \leq T_\mathcal{G}(G) \). Let \( Y \in \mathcal{R}_h \), \( X < Y \leq T_\mathcal{G}(G) \). There exists \( H \in Y \setminus X \). Hence \( H \in T_\mathcal{G}(G) \). According to Thm. 2.1, \( H \) can be constructed from a subset \( S \) of the class Hom Sub \( \{G\} \) by the operation \( \text{ext} \). Because \( H \) does not belong to \( X \), the set \( S \) must contain a linearly ordered group isomorphic to \( \Gamma_{\text{let}, i < j} G_i \) for some \( j \in I \). Then we have \( G \in Y \), whence \( Y = T_\mathcal{G}(G) \) and so (i) is valid. (iii) is a consequence of 2.1 and 2.3.

For each \( X \in \mathcal{R}_h \) we denote by \( a(X) \) the collection of all \( Y \in \mathcal{R}_h \) such that \( Y \) covers \( X \) in the lattice \( \mathcal{R}_h \).

From 3.6 we immediately obtain:

3.7. Corollary. Let \( X \in \mathcal{G}_1, \ X \neq R_0 \). Then there exists \( Y \in a(X) \cap \mathcal{R}_h \) such that \( Y \notin \mathcal{G}_1 \).

The proof of the following proposition will be omitted (it can be established by using similar arguments as in the proof of 3.6).

3.8. Proposition. Let \( X \in \mathcal{G}_1, \ X \neq R_0 \). Let \( I \) be as in 3.6 and let \( G = \Gamma_{\text{let}} G_i \), where each \( G_i \) belongs to \( A_0 \cap X \). Then the following conditions are equivalent:

(i) \( T_\mathcal{G}(G) \) covers \( X \);
(ii) for each \( G' \in A_0 \cap X \) and each \( j \in I \) there is \( i \in I \) such that \( i < j \) and \( G' \) is isomorphic to \( G_i \).

4. PRINCIPAL ELEMENTS OF \( \mathcal{R}_h \)

4.1. Proposition. Let \( X, Y \in \mathcal{R}_h, \ X \leq Y \). Assume that \( Y \) is a principal element of \( \mathcal{R}_h \). Then \( X \) is principal as well.

Proof. Let \( Y = T_\mathcal{R}(G) \). In view of 2.1, \( Y = \text{ext} \text{Hom Sub} \{G\} \). There exists a set \( S = \{H_i\}_{i \in I} \) of linearly ordered groups such that \( S \subset \text{Hom Sub} \{G\} \) and for each
4.2. Proposition. Let \( I \) be a nonempty set and for each \( i \in I \) let \( X_i \) be a principal element of \( \mathcal{R}_h \). Then \( X = \bigvee_{i \in I} X_i \) is a principal element of \( \mathcal{R}_h \) as well.

Proof. There are \( G_i \in \mathcal{G} \) such that \( X_i = T_h(G_i) \). We clearly have \( X = T_h(\{G_i\}_{i \in I}) = \bigvee \text{Hom Sub } \{G_i\}_{i \in I} \). There is a set \( S = \{H_j\}_{j \in J} \subseteq \mathcal{G} \) such that (i) \( S \subseteq \text{Hom Sub } \{G_i\}_{i \in I} \), and (ii) for each \( G_i \in \text{Hom Sub } \{G_i\}_{i \in I} \) there is \( j \in J \) having the property that \( G_i \) is isomorphic to \( H_j \). Again, we can assume that \( J \) is well-ordered.

Put \( H = \Gamma_{j \in J} H_j \). It is easy to verify that \( X = T_h(H) \), hence \( X \) is principal.

Let \( \alpha \) be a cardinal. We denote by \( l(\alpha) \) the first ordinal having the property that the set of all ordinals less than \( l(\alpha) \) has the cardinality \( \alpha \). Let \( J(\alpha) \) be the linearly ordered set dual to \( l(\alpha) \).

Let \( G \in \mathcal{G} \), \( G \neq \{0\} \). We put

\[
G(\alpha) = \Gamma_{j \in J(\alpha)} G_j,
\]

where each \( G_j \) is isomorphic to \( G \).

4.3. Lemma. Let \( G \in \mathcal{G} \), \( G \neq \{0\} \), \( \alpha > \text{card } G \). Then \( T_h(G) < T_h(G(\alpha)) \).

Proof. We have \( G \in \text{Hom } \{G(\alpha)\} \), hence \( T_h(G) \leq T_h(G(\alpha)) \). In view of 2.5, \( G(\alpha) \notin \mathcal{R}_h \). Hence \( T_h(G) < T_h(G(\alpha)) \).

4.4. Corollary. The class \( \mathcal{R}_{hp} \) has no maximal element. In particular, \( \mathcal{G} \) does not belong to \( \mathcal{R}_{hp} \).

Let \( G \in \mathcal{G} \), \( G \neq \{0\} \). In view of 4.3 there is a least cardinal \( \beta = \beta(G) \) such that \( T_h(G) < T_h(G_{(\beta(G))}) \).

The following proposition shows that there are many prime intervals in the lattice \( \mathcal{R}_h \).

4.5. Proposition. Let \( G \in \mathcal{G} \), \( G \neq \{0\} \). Then \( T_h(G) \) is covered by \( T_h(G_{(\beta(G))}) \) in the lattice \( \mathcal{R}_h \).

Proof. We have \( T_h(G) < T_h(G_{(\beta(G))}) \). Let \( X \in \mathcal{R}_h \), \( T_h(G) < X \leq T_h(G_{(\beta(G))}) \). There exists \( G_1 \in X \setminus T_h(G) \). Then \( G_1 \in \text{ext Hom Sub } \{G_{(\beta(G))}\} \). Hence there exists a set \( S \subseteq \text{Hom Sub } \{G_{(\beta(G))}\} \) such that \( G_1 \) can be constructed by means of \text{ext} from the set \( S \). In view of \( G_1 \notin T_h(G) \) there is \( H \in S \) such that \( H \) does not belong to \( \text{Hom Sub } \{G_{(\beta(G))}\} \). Therefore, from the construction of \( G_{(\beta(G))} \) it follows that there is a convex
subgroup $H_1$ of $H$ such that $H_1$ is isomorphic to $G(\mathcal{R}(G))$. Since $H_1 \in X$ we obtain $G(\mathcal{R}(G)) \in X$, implying $X = T_h(G(\mathcal{R}(G)))$.

From 4.5 and 3.1 we infer:

4.6. Corollary. Let $G \in \mathcal{R}_{hp}$. Then $a(T_h(G)) \cap \mathcal{R}_{hp} \neq \emptyset$.

Let $\mathcal{P}$ be the class of all prime intervals of the lattice $R_h$. From 4,5 and 4.2 we obtain:

4.7. Proposition. $\mathcal{P}$ is a proper collection.

References


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