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1. G. G. Hamedani [1] proved under suitable assumptions that the equation
\[ y'(t) = f(t, y(h_1(t)), \ldots, y(h_n(t)), y'(h_{n+1}(t)), \ldots, y'(h_{n+m}(t)), \lambda) \]
has exactly one solution defined in an interval \( J = (-\alpha, \alpha) \) and fulfilling an initial condition \( y(0) = \eta \), and this solution depends continuously on the parameter \( \lambda \).

In this note we use a contraction principle (given in Sec. 2 as Proposition) to establish the well-posedness of the Cauchy problem for the above type functional-differential equation with \( f, h, \lambda \) and \( \eta \) in certain \( \mathfrak{P}* \)-spaces (see e.g. [2]) which arise in a natural way. We shall treat the case \( n = m = 1 \) and \( J = (-\alpha, \alpha) \) with \( 0 < \alpha \leq \infty \) \( 0 \leq t_i, h(t_i), |h_i(t)| \leq t, t \in J \), the proof is similar and the reader can repeat it himself.

2. Let \( E \) be a Frechet space with a saturated sequence \( p_1, p_2, \ldots \) of seminorms which generates the topology of \( E \) (see e.g. [5]). Let \( A \) be a nonempty subset of \( E \) and let \( T \) be a one-to-one transformation of \( A \) into \( E \) for which \( T[A] \) is a closed set. Suppose that \( F_n \) \( (n = 1, 2, \ldots) \) and \( F_0 \) are mappings from \( A \) into \( E \) satisfying the following conditions:

(1) \( F_n[A] \subset T[A] \) for all \( n \geq 1 \), (2) \( \lim_{n \to \infty} F_n x = F_0 x \) for all \( x \) in \( A \), and (3) \( p_i(F_n x - F_n y) \leq k \cdot p_i(T x - T y) \) for all \( i \geq 1, n \geq 1 \) and \( x, y \) in \( A \), where \( 0 \leq k < 1 \).

Now, we give the following result of the type of Banach contraction principle:

**Proposition** ([3], [4]). Under the above assumptions there exists a unique point \( x_m \) \( (m = 0, 1, \ldots) \) in \( A \) such that \( F_m x_m = T x_m \), and \( T x_n \to T x_0 \) as \( n \to \infty \).

3. Throughout this part, \( J = [0, \infty) \), \( R \) is the Euclidean space, and \( C(J) \) denotes the set of all continuous real functions defined on \( J \).

The set \( C(J) \) let be considered as a vector space with the topology of almost uniform convergence (i.e., uniform convergence on compact subsets of \( J \)). This topology
is determined by the sequence \((p_n)\) of seminorms given as \(p_n(x) = \sup_{0 \leq t \leq n} |x(t)|\) for \(x\) in \(C(J)\), and therefore \(C(J)\) is a Fréchet space.

Let \(K\) and \(L \leq 1\) be nonnegative constants, and let \(G\) be a locally bounded function of \(J\) into itself. Next, we use the following notation:

\(\mathcal{G}\) — the set of all continuous real functions \(f\) defined on \(J \times R \times R \times R\) such that \(|f(t, x_1, y_1, \lambda) - f(t, x_2, y_2, \lambda)| \leq K|x_1 - x_2| + L|y_1 - y_2|\) for \(t \geq 0\) and \(x_1, x_2, y_1, y_2, \lambda\) in \(R\);

\(\mathcal{G}_0\) — the set of all \(f\) in \(\mathcal{G}\) such that \(|f(t, x, y, \lambda_1) - f(t, x, y, \lambda_2)| \leq G(t)|\lambda_1 - \lambda_2|\) for \(t \geq 0\) and \(x, y, \lambda_1, \lambda_2\) in \(R\);

\(\mathcal{U}\) — the set of all continuous functions \(\varphi\) of \(J\) into itself with \(\varphi(t) \leq t\) for \(t \geq 0\).

By (PC) we shall denote the problem of finding the solution on the half-line \(t \geq 0\) of the differential equation

\[
y'(t) = f(t, y(g(t)), y'(h(t)), \lambda)
\]
satisfying the initial condition

\[
y(0) = \eta;
\]
here \(f \in \mathcal{G}\), \(g\) and \(h\) in \(\mathcal{U}\), and \(\lambda, \eta\) in \(R\) are given. Obviously, our (PC) problem is equivalent to the equation

\[
x(t) = f \left( t, \eta + \int_0^{g(t)} x(s) \, ds, \, x(h(t)), \, \lambda \right)
\]
in the space \(C(J)\).

**Theorem.** For an arbitrary \(f \in \mathcal{G}\), \(g \in \mathcal{U}\), \(h \in \mathcal{U}\), \(\lambda \in R\) and \(\eta \in R\) there exists a unique function \(y_{(f, g, h, \lambda, \eta)}\) satisfying the (PC) problem on \(J\).

Assume, moreover, that the sets \(\mathcal{G}_0\), \(\mathcal{U}\) are given the \(\mathcal{U}\)-space structures ([2]) by the almost uniform convergence on \(J \times R \times R \times R\) and \(J\), respectively. Then the transformation

\[
(f, g, h, \lambda, \eta) \mapsto y_{(f, g, h, \lambda, \eta)}
\]
maps continuously the \(\mathcal{U}\)-product ([2]) \(\mathcal{G}_0 \times \mathcal{U} \times \mathcal{U} \times R \times R\) into \(C(J)\).

**Proof.** Let \(r > 0\) be a constant such that \(r^{-1}K + L < 1\). Let \(\psi = (f, g, h, \lambda, \eta) \in \mathcal{G} \times \mathcal{U} \times \mathcal{U} \times R \times R\). Define:

\[
(Tx)(t) = \exp(-rt)x(t),
\]
\[
(Fx)(t) = \exp(-rt)f \left( t, \eta + \int_0^{g(t)} x(s) \, ds, \, x(h(t)), \, \lambda \right)
\]
for \(x\) in \(C(J)\). Then \(F[C(J)] \subseteq C(J) = T[C(J)]\). For a positive integer \(n\) and \(u, v\) in \(C(J)\) and \(0 \leq t \leq n\), we have

\[
\left| f \left( t, \eta + \int_0^{g(t)} u(s) \, ds, \, u(h(t)), \, \lambda \right) - f \left( t, \eta + \int_0^{g(t)} v(s) \, ds, \, v(h(t)), \, \lambda \right) \right| \leq \]

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\[
\begin{align*}
&\leq K \int_{0}^{\eta(t)} |u(s) - v(s)| \, ds + L |u(h(t)) - v(h(t))| \\
&\leq K \int_{0}^{1} \exp(rs) ((Tu)(s) - (Tv)(s)) \, ds + L \exp(rh(t)) |(Tu)(h(t)) - (Tv)(h(t))| \\
&\leq \left( \int_{0}^{1} \exp(rs) \, ds + L \exp(rt) \right) p_n(T_u - T_v) \leq (r^{-1}K + L) \exp(rt) p_n(T_u - T_v),
\end{align*}
\]

and it follows that \( p_n(F_u - F_v) \leq (r^{-1}K + L) p_n(T_u - T_v). \) Consequently, Proposition is applicable to the mappings \( T, F \) and the space \( C(J) \). We conclude that there exists a unique \( x^* \) in \( C(J) \) and

\[
x^*_\eta(t) = f(t, \eta + \int_{0}^{\eta(t)} x^*_\eta(s) \, ds, x^*_\eta(h(t)), \lambda) \quad \text{for} \quad t \geq 0,
\]

which proves the first part of our result.

Let \( \psi_m = (f_m, g_m, h_m, \lambda_m, \eta_m) \in \mathcal{F}_0 \times \mathcal{U} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \) for \( m = 0, 1, \ldots \). Assume that \( \lim_{n \to \infty} f_n = f_0, \lim_{n \to \infty} g_n = g_0, \lim_{n \to \infty} h_n = h_0, \) and \( | \lambda_n - \lambda_0 | \to 0 \) and \( | \eta_n - \eta_0 | \to 0 \) as \( n \to \infty \). Further, let \( I = [0, a] \) be a compact subset of \( J \). We prove that \( \sup_{t \in I} |y_{\psi_n}(t) - y_{\psi_0}(t)| \to 0 \) as \( n \to \infty \).

Denote by \( C(I) \) the Banach space of all continuous real functions on \( I \) with the usual supremum norm \( \| \cdot \| \). Now, let us denote by \( T, F_m \) \((m = 0, 1, \ldots)\) the mappings on \( C(I) \) defined as above whenever \( f = f_m, g = g_m, h = h_m, \lambda = \lambda_m, \eta = \eta_m \) and \( x \in C(I) \). Obviously, \( F_m[C(I)] \subset C(I) = T[C(I)] \) and \( \|F_n u - F_n v\| \leq (r^{-1}K + L) \|u - v\| \).

Since \( Tu - Tv \) for \( n \geq 1 \) and \( u, v \in C(I) \). Moreover, for \( n \geq 1 \) and \( x \in C(I) \) we obtain

\[
\|(F_n x)(t) - (F_0 x)(t)\| \leq K |\eta_n - \eta_0| + K \left| \int_{0}^{\eta_0(t)} x(s) \, ds - \int_{0}^{\eta_0(t)} x(s) \, ds \right| + L |x(h_n(t)) - x(h_0(t))| + G(t) |\lambda_n - \lambda_0| + \left| f_n \left( t, \eta_0 + \int_{0}^{\eta_0(t)} x(s) \, ds, x(h_0(t)), \lambda_0 \right) - f_0 \left( t, \eta_0 + \int_{0}^{\eta_0(t)} x(s) \, ds, x(h_0(t)), \lambda_0 \right) \right|
\]

for \( t \) in \( I \). So we have \( \|F_n x - F_0 x\| \to 0 \) as \( n \to \infty \).

Finally, by our Proposition there exists a unique \( x^*_m \in C(I) \) \((m = 0, 1, \ldots)\) such that \( x_{\psi_m|I} = x^*_m \) and \( \|x_n - x_0\| \to 0 \) as \( n \to \infty \). This completes the proof of the theorem.

References


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