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# SOME REMARKS ON THE AUTOMORPHISM GROUP OF A COMPACT GROUP ACTION

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## 1. $H^s$-AUTOMORPHISMS

Let $M$ be a compact $n$-dimensional manifold without boundary and $G$ a compact Lie group which acts in a transitive way on $M$. Let $\mu : G \times M \to M$ be the group action, and let $\mu_a(m) \overset{\text{def}}{=} \mu(a, m)$, for each $a \in G$, $m \in M$.

We denote by $\mathscr{D}^s(M)$ the space of all diffeomorphisms of $M$ of Sobolev class $H^s$, i.e., $f \in \mathscr{D}^s(M)$ if and only if $f$ is bijective and $f, f^{-1} : M \to M$ are of class $H^s$. It is known [2], [3] that $\mathscr{D}^s(M)$ is a topological group and for $s > (n/2) + 1$ it is a Hilbert manifold whose tangent space at $e$, the identity of $\mathscr{D}^s(M)$, is given by $T_e(M) = \mathscr{X}^s(M) =$ the space of all $H^s$-vector fields on $M$. Moreover, if $X$ is in $\mathscr{X}^s(M)$. $s > (n/2) + 1$, and $\{f_t\}$ is its flow, then $f_t$ is a $C^1$ curve in $\mathscr{D}^s(M)$. More details and proofs can be found in [2] and [3].

Throughout the paper we suppose that $s > (n/2) + 1$, and that the Riemannian metric on $M$ is invariant under the action of $G$.

**1.1. Definition** ([2]). An automorphism of class $H^s$ is a diffeomorphism $f : M \to M$ of class $H^s$ which is $G$-invariant.

We denote by $\mathscr{D}^s_G(M)$ the group of all $H^s$-automorphisms of the action, namely:

$$\mathscr{D}^s_G(M) = \{f \in \mathscr{D}^s(M) \mid \forall a \in G,\ \mu_a \circ f = f \circ \mu_a\} .$$

**1.2. Remark.** It is not difficult to see that $\mathscr{D}^s_G(M)$ is a subgroup of $\mathscr{D}^s(M)$ and also a $C^\infty$-manifold, whose tangent space at $e$ is given by $T_e\, \mathscr{D}^s_G(M) = \{X \in \mathscr{X}^s(M) \mid X$ commutes with all infinitesimal generators of $\mu\}$.

## 2. $G$-INVARIANT FORMS OF CLASS $H^s$

Let $H^s A^p(M)$ be the space of $p$-differential forms on $M$ endowed with the $H^s$-topology, i.e., with the topology given by the inner product

$$(\alpha, \beta)_s = \int_M \alpha \wedge *\beta + \int_M (d + \delta)^s\, \alpha \wedge *(d + \delta)^s \beta ,$$

where $\alpha, \beta \in H^s A^p(M)$, $* : H^s A^p(M) \to H^s A^{n-p}(M)$ is the "star" Hodge operator and $\delta : H^s A^p(M) \to H^{s-1} A^{p-1}(M)$ is the adjoint of $d$ with respect to the inner product $(\cdot)_s$, given on $p$-forms by

$$\delta \overset{\text{def}}{=} (-1)^{(p+1)+1} * d *^{-1} .$$

The action of $G$ on $M$ determines is a natural way an action of $G$ on $H^s A^p(M)$, and we have

**2.1. Definition.** $\omega \in H^s A^p(M)$ is *G-invariant* iff for each $a \in G$, the following equality holds:

$$\mu_a^* \omega = \omega .$$

We denote by $H^s A_G^p(M)$ the space of $G$-invariant $p$-forms on $M$ of class $H^s$.

As in the non-equivariant case, there is a natural relation between $\mathscr{X}_G^s(M)$ and $H^s A_G^1(M)$:

**2.2. Proposition.** *There is an isomorphism between* $\mathscr{X}_G^s(M)$ *and* $H^s A_G^1(M)$.

Proof. Let $X$ be a $G$-invariant $H^s$-vector field on $M$. Then we can associate with $X$ an l-form $\tilde{X}$, where

$$\tilde{X}(Y) \overset{\text{def}}{=} g(X, Y) ,$$

$Y \in \mathscr{X}^s(M)$ and $g$ is the Riemannian structure on $M$. Since this correspondence is a bijective one, it is enough to verify that $X$ is $G$-invariant. For each $a \in G$, we succesively have

$$\mu_a^* \tilde{X}(Y) = \tilde{X}(\mu_a' Y) = g(X, \mu_a' Y) = g(\mu_a' X, \mu_a' Y) = \\ = \mu_a^* g(X, Y) = g(X, Y) = \tilde{X}(Y) ,$$

and then $\tilde{X} \in H^s A_G^1(M)$. q.e.d.

**2.3. Theorem** (*G-invariant version of Hodge decomposition theorem*). *Let* $\omega \in H^s A_G^p(M)$. *Then there are* $\alpha \in H^{s+1} A_G^{p-1}(M)$, $\beta \in H^{s+1} A_G^{p+1}(M)$, $\gamma \in C^\infty A_G^p(M)$, $\Delta\gamma = 0$, *such that*

$$\omega = \mathrm{d}\alpha + \delta\beta + \gamma .$$

*Furthermore,* $d\alpha$, $\delta\beta$ *and* $\gamma$ *are* $H^s$*-orthogonal and hence uniquely determined.*

Proof. By the classical Hodge theorem [2], there are $\alpha \in H^{s+1} A^{p-1}(M)$, $\beta \in H^{s+1} A^{p+1}(M)$ and $\gamma \in C^\infty A^p(M)$, $\Delta\gamma = 0$, such that

$$\omega = d\alpha + \delta\beta + \gamma .$$

Now using the invariance of the metric under the action of $G$ and Wattson's theorem [5] we conclude that $\alpha$, $\beta$, $\gamma$ are $G$-invariant.

q.e.d.

**2.4. Remark.** We can also answer the following question. Given a $G$-invariant $p$-form $\omega$, under what conditions is there a $G$-invariant $p$-form $\eta$ such that the equation

$$\Delta\eta = \omega$$

is satisfied? The answer is : if and only if

$$(\gamma, \omega)_0 = 0$$

for every $G$-invariant harmonic form $\gamma$.

Indeed, suppose that $\omega = \Delta\eta$ and $\gamma$ is harmonic. Then

$$(\gamma, \omega)_0 = (\gamma, \Delta\eta)_0 = (\Delta\gamma, \eta)_0 = (0, \eta)_0 = 0 .$$

On the other hand, suppose $\omega$ is a $G$-invariant form satisfying $(\gamma, \omega) = 0$ for each $G$-invariant harmonic form $\gamma$. From the decomposition

$$\omega = d\alpha + \delta\beta + \gamma$$

we have, using the particular $\gamma$ which is part of $\omega$,

$$\begin{aligned} 0 = (\gamma, \omega)_0 &= (\gamma, d\alpha)_0 + (\gamma, \delta\beta)_0 + (\gamma, \gamma)_0 = \\ &= (\delta\gamma, \alpha)_0 + (d\gamma, \beta)_0 + (\gamma, \gamma)_0 = (\gamma, \gamma)_0 ; \end{aligned}$$

hence $\gamma = 0$, $\omega = d\alpha + \delta\beta$.

We set $\eta = \mu + \gamma$ and try to solve $\Delta\mu = d\alpha$, $\Delta\gamma = \delta\beta$ separately. First we take

$$\Delta\mu = d\alpha .$$

Decomposing $\alpha$ we have

$$\begin{aligned} \alpha &= d\alpha_1 + \delta\beta_1 + \gamma_1 , \\ d\alpha &= d\delta\beta_1 . \end{aligned}$$

Further,

$$\begin{aligned} \beta_1 &= d\alpha_2 + \delta\beta_2 + \gamma_2 , \\ d\delta\beta_1 &= d\delta d\alpha_2 = (d\delta + \delta d)(d\alpha_2) = \Delta(d\alpha_2) , \\ d\alpha &= \Delta\mu \quad \text{with} \quad \mu = d\alpha_2 . \end{aligned}$$

We find $\gamma$ similarly.

For us, one of the most important consequences of the above theorem is the following

**2.5. Proposition.** *Let $X$ be a $G$-invariant $H^s$-vector field on $M$. Then there are a unique $G$-invariant divergence free $H^s$-vector field $Y$ and a $G$-invariant gradient $H^s$-vector field* grad $(p)$ *such that*

$$X = Y + \operatorname{grad}(p) .$$

*Moreover, when setting $P(X) \overset{\text{def}}{=} Y$, $P$ is a bounded linear operator from $\mathscr{X}^s_G(M)$ to the space of $G$-invariant vector fields with free divergence.*

Proof. The first part is an immediate consequence of the above propostion. Indeed, in terms of the corresponding $G$-invariant l-form $\tilde{X}$ we write

$$\tilde{X} = d\alpha + \delta\beta + \gamma$$

and set $d\alpha = dp$, $\tilde{Y} = \delta\beta + \gamma$. Since $\delta^2 = 0$, $\delta\tilde{Y} = 0$ it follows that $\operatorname{div} Y = 0$. For the second part we can use the general technique of Ebin-Marsden [2].

q.e.d.

We close this section with a proposition which will be very useful in the next section:

**2.6. Proposition.** *Let $\omega$ be a $G$-invariant volume form on $M$. Then we have the isomorphism*

$$\{X \lrcorner \omega \mid X \in \mathscr{X}^s_G(M)\} \simeq H^{s+1} A^{n-1}_G(M).$$

Proof. By the classical theory of differential forms, for each $\alpha \in H^{s+1} A^{n-1}_G(M)$ there exists $X \in \mathscr{X}^s(M)$ such that $X \lrcorner \omega = \alpha$. Hence it is enough to prove that $X$ is $G$-invariant. But for each $a \in G$ we successively have

$$X \lrcorner \omega = \alpha = \mu^*_a\alpha = \mu^*_a(X \lrcorner \omega) = \mu'_a X \lrcorner \omega ,$$

and hence $X$ is $G$-invariant. q.e.d.

## 3. VOLUME PRESERVING AUTOMORPHISMS

Let $\omega$ be a $G$-invariant $H^s$-volume from on $M$ and $\mathscr{D}^s_{G,\omega}(M)$ the subgroup of $\mathscr{D}^s_G(M)$ of all $\omega$-preserving $H^s$-automorphisms:

$$\mathscr{D}^s_{G,\omega}(M) = \{f \in \mathscr{D}^s_G(M) \mid f^*\omega = \omega\}.$$

This group is important in the study of flows with various symmetries (e.g. a flow in $\mathbb{R}^3$ that is symmetric with respect to a given axis).

Now it is easy to see that $\mathscr{D}^s_{G,\omega}(M) = \mathscr{D}^s_\omega(M) \cap \mathscr{D}^s_G(M)$. Since this intersection is in general not transversal, it is not obvious that $\mathscr{D}^s_{G,\omega}(M)$ is a submanifold of $\mathscr{D}^s_G(M)$. However, we shall prove that this is the case. More precisely, we have

**3.1. Theorem.** *$\mathscr{D}^s_{G,\omega}(M)$ is a closed Hilbert submanifold of $\mathscr{D}^s_G(M)$.*

Proof. Using the $G$-invariant version of Hodge theorem we have that the cohomology class of $\omega$,

$$[\omega]_s = \omega + d(H^{s+1} A^{n-1}_G(M)),$$

is a closed affine subspace of $H^s A^n_G(M)$.

Define the map

$$\psi_G : f \in \mathscr{D}^{s+1}_G(M) \to \psi_G(f) \overset{\text{def}}{=} f^*(\omega) \in [\omega]_s .$$

It is easy to see that

$$\psi_G^{-1}(\omega) = \mathscr{D}^s_{G,\omega}(M),$$

and then $\mathscr{D}^s_{G,\omega}(M)$ is a $C^\infty$-submanifold of $\mathscr{D}^s_G(M)$ if we prove that $\psi_G$ is a submersion. To this end it is enough to observe that

$$Te\,\psi_G(X) = d(X \lrcorner\, \omega),$$

and hence $Te\,\psi_G$ is onto by Proposition 2.6.

q.e.d.

In the sequel we shall try to understand other topological properties of the group $\mathscr{D}^s_G(M)$. We begin with a $G$-invariant version of the Moser theorem:

**3.2. Theorem.** *Let* $\mathscr{V}^s_G(M) = \{\gamma \in H^s A^n_G(M) \mid \nu > 0, \int_M \nu = \int_M \omega\}$. *Then there is a map* $\chi_G$, $\chi_G : \mathscr{V}^s_G(M) \to \mathscr{D}^s_G(M)$ *such that*

$$\psi_G \circ \chi_G = \textit{identity}.$$

Proof. Let $\nu \in \mathscr{V}^s_G(M)$ and $\nu_t = t\nu + (1-t)\,\omega$, so that $\nu_t \in \mathscr{V}^s_G(M)$. Since $\int_M \nu = \int_M \omega$, we have $\omega - \nu = d\alpha$. Define $X_t$ by $X_t \lrcorner\, \nu_t = \alpha$. Then it is easy to see that $X_t \in \mathscr{X}^s_G(M)$. Let $\{\varphi_t\}$ be the flow of $X_t$. Since $X_t$ is $G$-invariant it follows that the flow $\{\varphi_t\}$ of $X_t$ is $G$-invariant so that $\varphi_t \in \mathscr{D}^s_G(M)$. Now, it is enough to take

$$\chi_G(\nu) = \varphi_1^{-1},$$

and we obtain the desired result.

q.e.d.

Using the classical observation of R. Palais [4] that the topology of $\mathscr{D}^s_G(M)$ and $\mathscr{D}_G(M)$ (i.e. the group of $C^\infty$-G-invariant diffeomorphisms of $M$) is the same, we can deduce from the above theorem the following result.

**3.3. Theorem.** (*G-invariant version of Omori theorem*). *$\mathscr{D}_G(M)$ is diffeomorphic to $\mathscr{D}_{G,\omega}(M) \times \mathscr{V}_G(M)$. In particular, $\mathscr{D}_{G,\omega}(M)$ is a deformation retract of $\mathscr{D}_G(M)$.*

Proof. For the proof it is enough to observe that $\phi$ defined by

$$\phi(f, \nu) \stackrel{\text{def}}{=} f \circ \chi_G(\nu)$$

gives the desired diffeomorphism.

q.e.d.

In [1] W. Curtis showed that if the action of $G$ is free, then there is a $G$-invariant spray on $\mathscr{D}_G(M)$. Using this result and the same technique as in [2] we can build a $G$-invariant spray on $\mathscr{D}_{G,\omega}(M)$. More precisely, we have

**3.4. Theorem.** *If $Z_G$ is a $G$-invariant spray on $\mathscr{D}_G(M)$, then $S_G(X) \stackrel{\text{def}}{=} TP(Z_G \circ X)$ is a $G$-invariant spray on $\mathscr{D}_{G,\omega}(M)$, where $P$ is defined as in Proposition* 2.5.

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