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V. Sathyabhama

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GENERALIZED LC-IDENTITY ON GD-GROUPOIDS

V. SATHYABHAMA, Waterloo

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Introduction. A generalized LC-identity [1, 2] is given by

$$A_1(A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z))) . \tag{1}$$

This functional equation, when all the functions (operations) A_i $(i = 1, 2, \ldots, 6)$ are quasigroups defined on the same non-empty set G is investigated in [2] and its general solution is obtained by reducing it to a simpler equation. If this equation is satisfied on non-empty sets G_i $(i = 1, 2, \ldots, 7)$, then each of the operations A_i $(i = 1, 2, \ldots, 6)$ can be regarded as a GD-groupoid in a natural way.

In this paper we find the general solution of equation (1) defined on GD-groupoids, in terms of a loop operation $(+)$ and an arbitrary mapping ψ such that ψ is a mapping into the left nucleus of the loop $(+)$.

Basic definitions and notaitons. *A loop* $G(\cdot)$ is a quasigroup with an identity. If the loop $G(\cdot)$ satisfies the identity

$$(x \,.\, (x \,.\, y)) \,.\, z = x \,.\, (x \,.\, (y \,.\, z)) , \quad \text{for all} \quad x, y, z \in G ,$$

it is called an *LC-loop* [1]. When the operation $(\cdot)$ is replaced by quasigroups A_i $(i = 1, 2, \ldots, 6)$ defined on G, we get the functional equation (1).

A *GD-groupoid* is an ordered quadruple $(G_1, G_2, G; A)$ involving three non-empty sets G_1, G_2, G and the mapping $A: G_1 \times G_2 \to G$ such that the equations $A(a, y) = c$ and $A(x, b) = c$ always have solutions in $y \in G_2$ and $x \in G_1$ respectively, for every $a \in G_1$, $b \in G_2$ and $c \in G$. When these solutions are unique, the GD-groupoid is called a *G-quasigroup*. Throughout this paper we denote the GD-groupoids simply by the operation involved in it.

A GD-groupoid $(G_1, G_2, G; A_1)$ is homotopic to another GD-groupoid $(H_1, H_2, H; A_2)$ if there exist three surjections $\alpha: G_1 \to H_1$, $\beta: G_2 \to H_2$ and $\gamma: G \to H$ such that $\gamma A_1(x, y) = A_2(\alpha x, \beta y)$, for every $x \in G_1$, $y \in G_2$ in this case the triple $[\alpha, \beta, \gamma]$ is called *a homotopy*.

The following notations are used.

$$L_i(a)\, y = A_i(a, y) , \quad R_i(b)\, x = A_i(x, b) , \quad (i = 1, 2, \ldots, 6) .$$

We consider the functional equation (1), namely $A_1(A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z)))$, for all $x \in G_1$, $y \in G_2$ and $z \in G_3$ where the operations ($i = 1, 2, \ldots, 6$) are the GD-groupoids $(G_5, G_3, G; A_1)$, $(G_1, G_4, G_5; A_2)$, $(G_1, G_2, G_4; A_3)$, $(G_1, G_7, G; A_4)$, $(G_1, G_6, G_7; A_5)$ and $(G_2, G_3, G_6; A_6)$. Further, we assume that A_6 is a G-quasigroup and $R_1(c)$: $G_5 \to G$, $L_5(a)$: $G_6 \to G_7$ and $L_4(a)$: $G_7 \to G$ are bijections for fixed $c \in G_3$ and $a \in G_1$ respectively.

Putting $x = a$ in equation (1), we get

$$(2) \qquad A_1(L_2(a)\, L_3(a)\, y, z) = L_4(a)\, L_5(a)\, A_6(y, z)\,.$$

Also, with $x = a$ and $z = c$ simultaneously in (1), we have

$$(3) \qquad R_1(c)\, L_2(a)\, L_3(a)\, y = L_4(a)\, L_5(a)\, R_6(c)\, y\,.$$

Since $L_4(a)$, $L_5(a)$, $R_6(c)$ and $R_1(c)$ are bijections, from (3), we see that $L_2(a)\, L_3(a)$ is a bijection for $x = a \in G_1$. Hence, from (2) and (1) we obtain

$$(4) \qquad L_4(a)\, L_5(a)\, A_6((L_2(a)\, L_3(a))^{-1}\, A_2(x, A_3(x, y)), z) = A_4(x, A_5(x, A_6(y, z)))\,.$$

Putting $z = c \in G_3$, it follows from (4) that

$$(5) \qquad L_4(a)\, L_5(a)\, R_6(c)\, ((L_2(a)\, L_3(a))^{-1}\, A_2(x, A_3(x, y))) = A_4(x, A_5(x, R_6(c)\, y))\,.$$

From (4) and (5) we have

$$(6) \qquad L_4(a)\, L_5(a)\, A_6(R_6(c)^{-1}\, L_5(a)^{-1}\, L_4(a)^{-1}\, A_4(x, A_5(x, R_6(c)\, y), z) = A_4(x, A_5(x, A_6(y, z)))\,.$$

Equation (6) could be rewritten as:

$$(7) \qquad L_4(a)\, L_5(a)\, A_6(R_6(c)^{-1}\, L_5(a)^{-1}\, L_4(a)^{-1}\, A_4(x, A_5(x, u)), z) = A_4(x, A_5(x, A_6(R_6(c)^{-1}\, u, z)))\,,$$

where $R_6(c)\, y = u \in G_6$.

Now let

$$(8) \qquad L_5(a)^{-1}\, L_4(a)^{-1}\, A_4(x, A_5(x, u)) = K(x, u)\,, \quad x \in G_1\,, \quad u \in G_6\,.$$

Then K is the mapping $G_1 \times G_6 \to G_6$. By means of (8), (7) becomes,

$$(9) \qquad A_6(R_6(c)^{-1}\, K(x, u), z) = K(x, A_6(R_6(c)^{-1}\, u, z))\,, \quad x \in G_1\,, \quad u \in G_6\,, \quad z \in G_3\,.$$

On G_6 define an operation $(+)$ as follows:

$$s + t = A_6(R_6(c)^{-1}\, s\,, \quad L_6(b)^{-1}\, t)\,, \quad \text{for every} \quad s, t \in G_6\,.$$

That is

$$A_6(y, z) = R_6(c)\, y + L_6(b)\, z\,. \tag{10}$$

For the element $s \in G_6$, there is only one element $y \in G_2$ such that $s = R_6(c)\, y$, because $R_6(c)$ is a bijection. A similar argument holds for $L_6(b)\, z$ also. Thus, the operation $(+)$ is well-defined on G_6. Further, we note from (10) that $G_6(+)$ is the homotopic image of the GD-groupoid A_6 and is itself a GD-groupoid [3]. Besides, since the equations $A_6(y, c) = d$ and $A_6(b, z) = d$ have unique solutions for $y \in G_2$ and $z \in G_3$ (since A_6 is a G-quasigroup) $G_6(+)$ is a quasigroup.

Next, we will show that $G_6(+)$ is a loop. That is, $G_6(+)$ has an identity. Putting $y = b$ in (10), we have $L_6(b)\, z = R_6(c)\, b + L_6(b)\, z$, which implies that $R_6(c)\, b$ is the left identity in $G_6(+)$, since for every $u \in G_6$, there is a unique $z \in G_3$ such that $L_6(b)\, z = u$. Similarly, by putting $z = c$ in (10), we get $R_6(c)\, y = R_6(c)\, y + L_6(b)\, c$, showing thereby that $L_6(b)\, c$ is the right identity in $G_6(+)$. Thus, $G_6(+)$, having a left and a right identity, has an identity namely $A_6(b, c)$ denoted by 0, and therefore $G_6(+)$ is a loop.

From (9) and (10) we have

$$K(x, u) + v = K(x, u + v)\,, \quad x \in G_1\,, \quad u \in G_6\,, \quad L_6(b)\, z = v \in G_6\,. \tag{11}$$

Put $u = 0$, the identity element in $G_6(+)$. Then from (11), we get

$$K(x, 0) + v = K(x, v)\,. \tag{12}$$

Let

$$K(x, 0) = \psi x\,, \quad \text{where } \psi \text{ is the mapping } \quad G_1 \to G_6\,. \tag{13}$$

Then, (11), (12) and (13) yield,

$$(\psi x + u) + v = \psi x + (u + v)\,, \tag{14}$$

where ψ is a map $G_1 \to G_6$ and $(+)$ is a loop operation defined on G_6 and hence ψx belongs to the left nucleus of $(+)$.

Equations (2) and (10) yield,

$$A_1(w, z) = L_4(a)\, L_5(a)\, (R_6(c)\, (L_2(a)\, L_3(a))^{-1}\, w + L_6(b)\, z)\,, \qquad w \in G_5\,, \quad z \in G_3\,. \tag{15}$$

From (5) and (8), using (3) and (12), we get

$$\begin{aligned} A_2(x, A_3(x, y)) &= L_2(a)\, L_3(a)\, R_6(c)^{-1}\, K(x, R_6(c)\, y)\,, \\ &= L_2(a)\, L_3(a)\, R_6(c)^{-1}\, (\psi x + R_6(c)\, y)\,. \end{aligned} \tag{16}$$

From (8) and (12), we have

$$(17) \qquad A_4(x, A_5(x, u)) = L_4(a)\, L_5(a)\, (\psi x + u)\,.$$

Putting $L_2(a)\, L_3(a) = \alpha$, $L_4(a)\, L_5(a) = \beta$, $R_6(c) = \gamma$, $L_6(b) = \delta$, equations (15), (16), (17) and (10) yield

$$(18) \qquad \begin{aligned} &A_1(w, z) = \beta(\gamma\alpha^{-1} w + \delta z)\,, \quad w \in G_5\,, \quad z \in G_3\,, \\ &A_2(x, A_3(x, y)) = \alpha\gamma^{-1}(\psi x + \gamma y)\,, \quad x \in G_1\,, \quad y \in G_2\,, \\ &A_4(x, A_5(x, u)) = \beta(\psi x + u)\,, \quad x \in G_1\,, \quad u \in G_6\,, \\ &A_6(y, z) = \gamma y + \delta z\,, \quad y \in G_2\,, \quad z \in G_3\,. \end{aligned}$$

Thus, we have proved part of the following theorem.

Theorem. *Let $(G_5, G_3, G; A_1)$, $(G_1, G_4, G_5; A_2)$, $(G_1, G_2, G_4; A_3)$, $(G_1, G_7, G; A_4)$, $(G_1, G_6, G_7; A_5)$ and $(G_2, G_3, G_6; A_6)$ be GD-groupoids satisfying the functional equation (1) and let $R_1(c)\colon G_5 \to G$, $L_5(a)\colon G_6 \to G_7$, $L_4(a)\colon G_7 \to G$ be bijections for fixed $c \in G_3$, $a \in G_1$. Further, let A_6 be a G-quasigroup. Then there exists a loop $(+)$ defined on the set G_6 and a mapping $\psi\colon G_1 \to G_6$ such that ψ is a mapping into the left nucleus of the loop $(+)$ and the general solution of equation (1) is given by (18) and conversely.*

The converse part of this theorem can esily be established by simply substituting (18) into (1) and taking into account that ψx belongs to the left nucleus of the loop $G_6, (+)$.

Now we will deduce the result proved in [2] from Theorem 1, that is let us consider the case when all the GD-groupoids A_i $(i = 1, 2, \ldots, 6)$ are quasigroups defined on the same set G. If we represent the quasigroups A_5 and A_4 as

$$(19) \qquad A_5(x, y) = C(x, y)\,,$$

and

$$(20) \qquad A_4(x, y) = \beta\, K(x, y)\,,$$

then C and K are quasigroups. From (5) we have

$$(21) \qquad A_2(x, A_3(x, y)) = \alpha\gamma^{-1}\, K(x, C(x, \gamma y))\,,$$

and, from (18),

$$(22) \qquad A_1(x, y) = \beta(\gamma\alpha^{-1} x + \delta y)\,, \quad A_6(x, y) = \gamma x + \delta y\,.$$

Substituting (19), (20), (21) and (22) into (1) and in the resulting equation replacing γy by y and δz by z, we get

$$(23) \qquad K(x, C(x, y)) + z = K(x, C(x, y + z))\,,$$

which is precisely the reduced equation (7) in [2]. Also, with $y = 0$, the identity of the loop $(+)$, and writing $K(x, C(x, 0)) = \psi(x)$, from (23) we obtain

$$\psi(x) + z = K(x, C(x, z)) . \tag{24}$$

From (23) and (24) we see that

$$(\psi(x) + y) + z = \psi(x) + (y + z) ,$$

which shows that $\psi(x)$ belongs to the left nucleus of the loop $G, (+)$, [2].

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Author's address: University of Waterloo, Waterloo, Ontario, Canada. N2L 3G1.