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SOME REMARKS ON DOMATIC NUMBERS OF GRAPHS

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E. J. Cockayne and S. T. Hedetniemi in the papers [1] and [2] define the domatic number of an undirected graph. Here we shall present some results concerning this concept. We shall investigate finite undirected graphs without loops and multiple edges.

A dominating set in a graph $G$ is a subset $D$ of the vertex set $V(G)$ of $G$ with the property that each vertex of $V(G) - D$ is adjacent to at least one vertex of $D$. A partition of $V(G)$ into dominating sets is called a domatic partition of $G$. The maximal number of classes of a domatic partition of a graph $G$ is called the domatic number of $G$ and is denoted by $d(G)$.

In [2] it is suggested to relate the domatic number of a graph $G$ to the connectivity of this graph. In this paper we shall prove some results concerning this topic.

The vertex (or edge) connectivity degree of a graph $G$ is the minimal cardinality of a subset of the vertex set (or the edge set, respectively) of $G$ with the property that by deleting this set from $G$ a disconnected graph is obtained. (To delete a subset of the vertex set of $G$ means to delete all vertices of this set and all edges which are incident to these vertices. To delete a subset of the edge set of $G$ means to delete only all edges of this set.) The vertex connectivity degree of $G$ will be denoted by $\omega(G)$, its edge connectivity degree by $\sigma(G)$.

**Theorem 1.** Let $p$ and $q$ be non-negative integers, $p < q$. Then there exists a graph $G$ such that $\omega(G) = p$, $d(G) = q$.

**Proof.** Take two copies $G'$, $G''$ of the complete graph $K_q$ with $q$ vertices. If $p = 0$, then $G$ is the graph whose connected components are $G'$ and $G''$. If $p \neq 0$, we choose pairwise distinct vertices $u_1, \ldots, u_p$ in $G'$ and $v_1, \ldots, v_p$ in $G''$ and identify $u_i$ with $v_i$ for each $i = 1, \ldots, p$. In the following we shall denote the vertex obtained by identifying $u_i$ with $v_i$ by $w_i$ for $i = 1, \ldots, p$. The remaining vertices of $G'$ (or $G''$) will be denoted by $u_{p+1}, \ldots, u_q$ (or $v_{p+1}, \ldots, v_q$, respectively). In the case $p = 0$ we denote the vertices of $G'$ by $u_1, \ldots, u_q$ and the vertices of $G''$ by $v_1, \ldots, v_q$. If we delete the set $\{w_1, \ldots, w_p\}$ from $G$, we obtain a disconnected graph. As each of the vertices $w_1, \ldots, w_p$ is adjacent to all the other vertices of $G$, after deleting less than $p$ vertices
Let $G$ be a graph with $p$ vertices and $q$ edges. We define $D_t = \{w_i\}$ for $i = 1, \ldots, p$ and $D_i = \{w_{i+1}\}$ for $i = p + 1, \ldots, q$. Evidently, $\sigma(G) = p$, where $G$ is the graph thus obtained. Taking $D_t = \{w_i\}$ for $i = 1, \ldots, p$ we obtain a domatic partition $\{D_1, \ldots, D_p\}$ and, as $\delta(G) = p$, we have $d(G) = p$.

**Theorem 2.** Let $p$ and $q$ be non-negative integers, $p < q$. Then there exists a graph $G$ such that $\sigma(G) = p$, $d(G) = q$.

**Proof.** We take again two copies of $K_2$. The vertices of one copy $u_1, \ldots, u_q$ (or $v_1, \ldots, v_q$, respectively). If $p = 0$, the graph $G$ is the same as in the proof of Theorem 1. If $p > 0$, we join $u_i$ with $v_i$ by an edge for each $i = 1, \ldots, p$. Evidently $\sigma(G) = p$, where $G$ is the graph thus obtained. Taking $D_t = \{u_i\}$ for $i = 1, \ldots, q$ we obtain a domatic partition $\{D_1, \ldots, D_q\}$ and, as $\delta(G) = q - 1$, we have $d(G) = q$.

**Theorem 3.** Let $h$ be a positive integer. Then there exists a graph $G$ such that $\omega(G) - d(G) = \sigma(G) - d(G) = h$.

**Proof.** Let $n = 2h + 4$ and consider the complete graph $K_n$. As $n$ is even, there is a linear factor $F$ of $K_n$. Let the edges of $F$ be $e_1, \ldots, e_{n+2}$, let $u_i$, $v_i$ be the endpoints of the edge $e_i$ for $i = 1, \ldots, h + 2$. Let $G$ be the graph obtained from $K_n$ by deleting all edges of $F$. Evidently each subset of $V(G)$ which induces a disconnected subgraph of $G$ is of the form $\{u_i, v_i\}$ for some $i$. Therefore $\omega(G) = n - 2 = 2h + 2$. It is easy to prove that also $\sigma(G) = n - 2 = 2h + 2$. No vertex of $G$ is adjacent to all the other vertices, therefore each dominating set of $G$ has at least two vertices. This implies $d(G) \leq n/2$. Putting $D_i = \{u_i, v_i\}$ for $i = 1, \ldots, h + 2$ we obtain a domatic partition of $G$ and hence $d(G) = n/2 = h + 2$. We have

$$\omega(G) - d(G) = \sigma(G) - d(G) = h.$$ 

The graph from the proof of Theorem 3 also has the property that $d(G) = \frac{1}{2} \delta(G) + 1$. We express a conjecture.

**Conjecture.** For each graph $G$ we have

$$d(G) \geq \frac{1}{2} \delta(G) + 1.$$ 

At the end we turn to another problem suggested in [2] — to characterize the uniquely domatic graphs.

A graph $G$ is called uniquely domatic, if there exists exactly one domatic partition of $G$ with $d(G)$ classes.

We shall characterize the uniquely domatic graphs whose domatic number is 2. First we prove a lemma.
**Lemma.** Each uniquely domatic graph with a domatic number at least 2 is connected.

**Proof.** Let $G$ be a disconnected graph with $d(G) \geq 2$. Then each connected component of $G$ has at least two vertices; otherwise the domatic number of $G$ would be 1. Let $d(G) = d$, let $\{D_1, \ldots, D_d\}$ be a domatic partition of $G$. Let $C$ be a connected component of $G$, let $V(C)$ be its vertex set. As each vertex of $C$ can be adjacent only to vertices of $C$, we have $D_i \cap V(C) \neq \emptyset$ for each $i = 1, \ldots, d$ and $\{D_1 \cap V(C), \ldots, D_d \cap V(C)\}$ is a domatic partition of $C$. Put $D_i = (D_1 - V(C)) \cup (D_2 \cap V(C)), D'_i = (D_2 - V(C)) \cup (D_1 \cap V(C)), D'_1 = D_1$ for $i = 3, \ldots, d$. It is easy to prove that $\{D'_1, \ldots, D'_d\}$ is a domatic partition of $G$ different from $\{D_1, \ldots, D_d\}$ and hence $G$ is not uniquely domatic.

**Theorem 4.** A graph with the domatic number 2 is uniquely domatic, if and only if it is a star or a complete graph $K_2$.

**Proof.** Let $G$ be a uniquely domatic graph with the domatic number 2. By Lemma the graph $G$ must be connected. If $G$ is neither a star nor $K_2$, then there exists a spanning tree $T$ of $G$ which is neither a star nor $K_2$. Therefore there exists an edge $e$ of $T$ which joins two non-terminal vertices of $T$. Let $T'$ and $T''$ be the connected components of the forest obtained from $T$ by deleting $e$. None of the graphs $T', T''$ is an isolated vertex, therefore $d(T') = d(T'') = 2$. Let $\{D'_1, D'_2\}$ (or $\{D''_1, D''_2\}$) be a domatic partition of $T'$ (or $T''$, respectively). It is easy to see that $\{D'_1 \cup D''_1, D'_2 \cup D''_2\}$ and $\{D'_1 \cup D''_1, D'_2 \cup D''_2\}$ are domatic partitions of $T$ and also of $G$. These partitions are evidently different, which is a contradiction with the assumption that $G$ is uniquely domatic. Therefore $G$ must be either a star or $K_2$. On the other hand, the unique domatic partition of a star into two classes is such that one class consists only of the center and the other consists of all other vertices, because if a terminal vertex of a star belonged to the same class as the center, it would not be adjacent to a vertex of the other class. An analogous situation occurs in the case of $K_2$.

**References**


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