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Graphs of semigroups

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Analogously to graphs of groups (see e.g. [1]) we shall introduce graphs of semigroups.

Let $S$ be a semigroup, let $A$ be its subset. The graph $G(S, A)$ is a directed graph whose vertices are elements of $S$ and in which there is a directed edge from a vertex $u$ into a vertex $v$ if and only if $v = ua$, where $a \in A$.

Here we shall characterize finite graphs $G(S, A)$, where $A$ is a one-element set. Thus we shall have $A = \{a\}$ and instead of $G(S, \{a\})$ we shall write simply $G(S, a)$. We shall admit loops and consider them as cycles of the length 1.

Every graph $G(S, a)$ has the property that the outdegree of each of its vertices is 1. The structure of such graphs is well-known. If such a graph is finite, then each of its connected components contains exactly one cycle (by a cycle we mean a directed circuit).

After deleting all edges of this cycle a forest is obtained. Each tree of this forest has the property that for each of its vertices there is a directed path going from this vertex to a vertex of the cycle (Fig. 1). If $C$ is a connected component of such a graph, then by $\pi(C)$ we denote the length of the cycle contained in $C$ (it may be 1, if this
cycle is a loop) and by $\lambda(C)$ we denote the maximal length of a directed path in this component which contains no edge of the cycle (it may be 0, if $C$ consists only of a cycle).

Now we shall prove a theorem which gives a characterization of the finite graphs $G(S, a)$.

**Theorem.** Let $G$ be a finite directed graph in which each vertex has the outdegree 1. The graph $G$ is isomorphic to the graph $G(S, a)$ for a semigroup $S$ and its element $a$ if and only if it contains a connected component $C$ with the property that for each connected component $D$ of $G$ the number $x(D)$ divides $x(C)$ and $\lambda(D) \leq \lambda(C) + 1$.

**Proof.** Suppose that $G$ is isomorphic to $G(S, a)$ for some $S$ and $a$. Let $a$ have a period $h$ and a pre-period $k$; this means that the elements $a, a^2, \ldots, a^{h+k-1}$ are pairwise distinct and $a^{h+k} = a^k$. Hence in $G$ there exists a cycle of the length $h$ and a directed path of the length $k$ whose terminal vertex belongs to this cycle; the initial vertex of this path corresponds to the element $a$. Now let $x$ be an arbitrary vertex of $G$ (i.e. an element of $S$); let $D$ be the connected component of $G$ containing $x$. Let $p$ be the length of the directed path outgoing from $x$, incoming into a vertex of a cycle and containing no edge of this cycle; evidently $p \leq \lambda(D)$. Let $q = x(D)$. Then the elements $x, xa, xa^2, \ldots, xa^{p+q-1}$ are pairwise distinct and $xa^{q} = xa^p$. If $q$ does not divide $h$, then $k$ and $h + k$ are not congruent modulo $q$ and thus $xa^k = xa^{h+k}$, which is a contradiction with the assumption $a^{h+k} = a^k$. Hence $q$ must divide $h$ and $h$ is $\lambda(C)$, where $C$ is the connected component of $G$ containing $a$. Now suppose $p \geq k + 2$. Then $xa^{k+1}$ is distinct from $xa^l$ for each $l \neq k + 1$. But, as $a^{h+k} = a^k$, we must have $xa^{k+1} = xa^{h+k+1}$, which is a contradiction. Hence $p \leq k + 1 \leq \lambda(C) + 1$. Thus the necessity of the condition is proved.

Now suppose that the condition is fulfilled. In $C$ take a directed path containing no edge of a cycle and having the length $\lambda(C)$; its initial vertex will be $a$. Take all sources of $G$ and if $G$ contains connected components distinct from $C$ which are cycles, choose one vertex in each of them. The set thus obtained will be denoted by $B$. The vertex $a$ and the vertices of $B$ will be considered elements of a semigroup $S$. Each remaining vertex will be denoted as a power of $a$ or a product of an element of $B$ with a power of $a$ in the way corresponding to the definition of $G(S, a)$. Further, we introduce the equality $xb = b$ for each $x \in S$ and each $b \in B$. Thus we have defined a semigroup $S$ such that $G$ is isomorphic to $G(S, a)$.

**Reference**


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