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ON HAMILTONIAN CIRCUITS AND SPANNING TREES OF HYPERCUBES

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1. INTRODUCTION

The aim of this paper is to prove that certain trees are spanning trees of the hypercubes $Q_n (n \geq 1)$. Obviously, the simplest spanning tree of $Q_n$ is the path $p_{2^n - 1}$ of length $2^n - 1$ (where the length of a path is measured by the number of its edges). Other two spanning trees of $Q_n$ (similar to each other) have been found by Nebeský when solving a different problem in [8]; they arise by means of certain “reduplication” of binary trees.

A complete solution of the problem of spanning trees of hypercubes would be provided by characterizing them; such a characterization seems to be of interest especially in view of the fact that hypercubes have been characterized (cf. e.g. [1], [6] and [7]). Unfortunately, we are not able to solve the above mentioned problem; we present it therefore as an open question (together with some related conjectures) at the end of the paper (Sec. 5).

In Sec. 2 we prove certain assertions concerning the structure and properties of hamiltonian circuits and paths in $Q_n$. Using them we find in Sec. 4 some spanning trees of $Q_n$. Sec. 3 describes spanning trees of $Q_n$ obtained in a different way, namely, by modifications of binary trees.

In the whole paper we deal only with finite undirected graphs without loops and multiple edges. $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. The maximum degree of vertices in $G$ will be denoted by $\text{maxdeg} (G)$.

The hypercube $Q_n (n \geq 1)$ is defined in the usual way (cf. e.g. [2]); its vertices are all the vectors of length $n$ consisting of 0’s and 1’s. For $u, v \in V(Q_n)$, $q(u, v)$ denotes the Hamming distance of $u$ and $v$, i.e., the number of coordinates in which $u$ and $v$ differ from each other. $(u, v) \in E(Q_n)$ iff $q(u, v) = 1$. Given $i, 1 \leq i \leq n$, $Q_n$ can be decomposed into two copies of $Q_{n-1}$ (denoted by $Q'_{n-1}$, $Q''_{n-1}$) whose vertices are joined by $2^{n-1}$ edges of a perfect matching; the vertices of $Q'_{n-1}(Q''_{n-1})$ are those of $Q_n$ with the $i$-th coordinate equal to 0 (1, respectively). We call this decomposition of $Q_n$ canonical (more precisely, $i$-canonical).
The notion of the so called $C_n$-valuation of a graph will be frequently used (cf. [5]); the definition and the basic property, modified for the case of trees, are as follows: a tree $T$ is said to be $C_n$-valued, if the edges of $T$ are labelled by integers from $\{1, \ldots, n\}$ in such a way that for any path $p$ of $T$ there is $k \in \{1, \ldots, n\}$ such that an odd number of edges of $p$ are assigned $k$. Then $T$ is isomorphic to a subgraph of $Q_n$ (in other words: $T$ is embeddable in $Q_n$) if and only if there is a $C_n$-valuation of $T$.

Given a $C_n$-valuation of $T$ and a path $p$ in $T$, we define "the odd set of $p$" by

$O(p) = \{k \in \{1, \ldots, n\}; \text{ an odd number of edges of } p \text{ are labelled by } k\}$.

With a $C_n$-valuation of $T$ a certain embedding of $T$ in $Q_n$ can be associated, i.e., an injection $\varepsilon: V(T) \to V(Q_n)$ such that $(u, v) \in E(T) \Rightarrow (\varepsilon(u), \varepsilon(v)) \in E(Q_n)$. The mapping $\varepsilon$ is obviously an isomorphism of $T$ to a subgraph of $Q_n$ (this subgraph not necessarily being an induced one). If $p$ is a path in $T$ with end-vertices $u, v$ and $|O(p)| = 1$, then $\varepsilon(\varepsilon(u), \varepsilon(v)) = 1$.

It is clear that every tree can be $C_n$-valued (for $n$ sufficiently large). By $\dim T$ we shall denote the smallest $n$ such that there is a $C_n$-valuation of $T$ (obviously, $\dim T$ is the smallest $n$ with the property that $T$ is isomorphic to a subgraph of $Q_n$).

We shall frequently need $C_n$-valuations of paths; let us construct one of them as follows: for $i \geq 1$ let $i = 2^j \cdot m$, where $m$ is odd. Putting $a_i = j + 1$ we obtain the sequence $\{a_i\}_{i \geq 1}$ whose members are

$1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5, 1, \ldots$

It is not difficult to see that for $k \geq 1$ the values $\{a_i\}_{i=1}^{2k-1}$ may be used as the values of a $C_k$-valuation of the path $p_{2k-1}$ of length $2^k - 1$. Let us call this valuation the basic $C_k$-valuation of $p_{2k-1}$. We have $O(p_{2k-1}) = \{k\}$ and the basic $C_k$-valuation may easily be modified so that e.g. $O(p_{2k-1}) = \{1\}$.

2. SOME STRUCTURAL PROPERTIES OF HAMILTONIAN CIRCUITS AND PATHS IN $Q_n$

In this section we derive certain properties of hamiltonian circuits and paths in $Q_n$ that will be needed in Sec. 4 (some of them seem to be of a certain interest by themselves). If $u, v \in V(Q_n)$, $u \neq v$, and if an arbitrary path $p$ containing $u$ and $v$ or a hamiltonian circuit $c$ in $Q_n$ is given, we use in addition to the well-known Hamming distance $d(u, v)$ of the vertices $u, v$ also the notion of "the distance of $u, v$ along $p$ or along the circuit $c$" (with the obvious meaning); the vertices $u, v$ have always two distances $d_1, d_2$ along $c$ (where $d_1 + d_2 = 2^n$ and the equality $d_1 = d_2$ may hold).

2.1. Proposition. Let $n \geq 2$, $u, v \in V(Q_n)$, $u \neq v$. Let $r \equiv d(u, v) \pmod{2}$, $d(u, v) \leq r \leq 2^n - d(u, v)$. Then there is a hamiltonian circuit $c$ in $Q_n$ such that one of the distances of $u, v$ along $c$ is $r$ (and the other is $2^n - r$).
Proof. Let $s, t, d$ and $n$ be positive integers. We shall write $HC(s, t; d, n)$ if the following holds: for any $u, v \in V(Q_n)$ fulfilling $q(u, v) = d$ there is a hamiltonian circuit in $Q_n$ such that one of the distances of $u$ and $v$ along this circuit is $s$ and the other is $t$. Obviously, $HC(s, t; d, n)$ iff $HC(t, s; d, n)$ and if $HC(s, t; d, n)$, then

1. $s + t = 2^n$,
2. $d \leq \min(s, t)$, $d \leq n$, and
3. $d \equiv s \equiv t \pmod{2}$.

We shall show now that $HC(s, t; d, n)$ holds for all quadruples $s, t, d, n$ of positive integers fulfilling (1), (2) and (3); obviously, this will prove the proposition.

First we prove two lemmas using the notion of a canonical decomposition of $Q_{n+1}$ (into $Q'_n$ and $Q''_n$).

**Lemma 1.** $HC(s, t; d, n) \Rightarrow HC(s, t + 2^n; d, n + 1)$.

The implication easily follows from Fig. 2.1; given $u, v \in V(Q_{n+1})$ with $q(u, v) = d$, it is always possible to find an $i$-canonical decomposition of $Q_{n+1}$ such that both $u$ and $v$ belong to the same part of it (e.g. to $Q'_n$). Then, $HC(s, t; d, n)$ guarantees the existence of a hamiltonian circuit $c'$ in $Q'_n$ such that the distances of $u'$ and $v'$ along $c'$ are $s$ and $t$. Let $c''$ be the image of $c'$ in $Q''_n$. A hamiltonian circuit $c$ of $Q_{n+1}$ with the properties required (i.e., such that the distances of $u$ and $v$ along $c$ are $s$ and $t + 2^n$) can now be easily constructed from $c'$ and $c''$ according to Fig. 2.1.

![Fig. 2.1.](image)

**Lemma 2.** $HC(s, t; d, n)$ and $0 \leq q < t \Rightarrow HC(s + 2q + 1, s + 2t - 2q - 1; d + 1, n + 1)$.

Using again a canonical decomposition of $Q_{n+1}$ we assume that if $u, v \in V(Q_{n+1})$, $q(u, v) = d + 1$, then $u \in V(Q'_n)$ and $v \in V(Q''_n)$. The construction then easily follows from Fig. 2.2.
We are now ready to prove the main proposition using induction on \( n \): both \( \text{HC}(1, 3; 1, 2) \) and \( \text{HC}(2, 2; 2, 2) \) obviously hold. Let \( n \geq 2 \), suppose \( \text{HC}(s', t'; d', n) \) holds whenever \( s' + t' = 2^n \), \( d' \leq \min(s', t') \), \( d' \leq n \) and \( s' \equiv t' \equiv d' \pmod{2} \). Let \( s + t = 2^{n+1} \), \( d \leq \min(s, t) \), \( d \leq n + 1 \), \( s \equiv t \equiv d \pmod{2} \). If \( d = 1 \), then 

\[ s + t; \text{let e.g. } s < t. \] Using Lemma 1 we have \( \text{HC}(s, t - 2^n; 1, n) \Rightarrow \text{HC}(s, t; 1, n + 1) \). Let \( d > 1 \); then \( 1 \leq d - 1 \leq n \) and therefore \( \text{HC}(d - 1, 2^n - d + 1; d - 1, n) \) holds. Suppose \( s \leq t \) and put \( q = (s - d)/2 \). Then \( 0 \leq q < 2^n - d + 1 \) and, using Lemma 2, we obtain \( \text{HC}(s, t; d, n + 1) \), q.e.d.

The following result is an easy consequence of 2.1.

2.2. **Corollary.** Let \( n \geq 1 \), consider the path \( p_{2^n-1} \) of length \( 2^n - 1 \). Assume \( i, j \in \{1, \ldots, n\}, i \neq j, 1 \leq l \leq 2^n - 1 \). Let \( p_l \) be the initial part of \( p_{2^n-1} \) of length \( l \). Then it is possible to construct a \( C_n \)-valuation of \( p_{2^n-1} \) such that \( O(p_l) = \{i\} \) if \( l \) is odd and \( O(p_l) = \{i, j\} \) if \( l \) is even.

In fact, if e.g. \( l \) is odd, choose \( u, v \in V(Q_n) \) differing in the \( i \)-th coordinate; then \( \varrho(u, v) = 1 \) and there is a hamiltonian circuit \( c \) in \( Q_n \) such that one of the distances of \( u, v \) along \( c \) is \( l \). Let us delete the edge incident with \( u \) from the other chord (of length \( 2^n - l \)) of \( c \). In this way a hamiltonian path \( p \) in \( Q_n \) is obtained; since \( p \) is embedded in \( Q_n \), we obviously can use the corresponding \( C_n \)-valuation of \( p \) as the desired one and proceed quite similarly also in the case of even \( l \).

For \( u \in V(Q_n) \) let \( \bar{u} \) denote the vertex opposite to \( u \) in \( Q_n \) (i.e. such that \( \varrho(u, \bar{u}) = n \)).

2.3. **Proposition.** Let \( u, v \in V(Q_n), \varrho(u, v) \equiv 1 \pmod{2} \). Then there is a hamiltonian path \( p \) in \( Q_n \) with end-vertices \( u \) and \( v \). Moreover, if \( u \neq \bar{v} \) (i.e. if \( \varrho(u, v) < n \)), then \( p \) can be constructed in such a way that the distance of \( u \) and \( v \) along \( p \) equals...
Proof. 1. Assume first \( q(u, v) = 1 \); according to 2.1 there is a hamiltonian circuit \( c \) in \( Q_n \) such that one of the distances of \( v \) and \( \bar{v} \) along \( c \) is \( n \). Without loss of generality we may assume that the edge \((u, v)\) belongs to the chord of length \( n \) of \( c \) joining \( v \) and \( \bar{v} \). Removing the edge \((u, v)\) from \( c \) we obtain the required hamiltonian path.

2. Assume now \( 3 \leq q(u, v) \leq n \). Let, without loss of generality, \( u = (0, \ldots, 0) \), \( v = (0, \ldots, 0, 1, \ldots, 1) \), put \( w = (0, \ldots, 0, 1, 0) \). Let \( Q'_n \) and \( Q''_n \) be parts of the \( n \)-canonical decomposition of \( Q_n \) (i.e., \( Q'_n \) is the hypercube induced in \( Q_n \) by all the vertices having 0 as its \( n \)-th coordinate). It follows from what has been proved above that there is a hamiltonian path \( p' \) in \( Q'_n \) joining \( u \) and \( w \) such that the distance of the vertices \((0, \ldots, 0)\) and \((1, 1, \ldots, 1, 0, 0)\) along \( p' \) is \( n - 2 \); we may assume that the beginning of \( p' \) is formed by the vertices \((0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, 0, \ldots, \ldots, 0), \ldots, (1, 1, \ldots, 1, 0, 0)\). Let us extend \( p' \) by adding the edge joining \( w = (0, \ldots, 0, 1, 0) \) with \( w'' = (0, \ldots, 0, 1, 1) \), where \( w'' \) belongs to \( Q''_n \). We have \( q(w'', v) = q(u, v) - 2 \) and the distance of \( v \) and \( w'' \) in \( Q''_n \) is again odd, therefore by induction there is a hamiltonian path \( p'' \) in \( Q''_n \) with end-vertices \( v \) and \( w'' \). Joining the paths \( p' \) and \( p'' \) by the edge \((w, w'')\) we obtain a path \( p \) with the desired properties, q.e.d.

2.4. Remark. From 2.3 the following fact can be easily obtained: Let \( u, v \in V(Q_n) \), \( q(u, v) \equiv 1 \, (\text{mod } 2) \); let \( l_1, l_2 \) be integers fulfilling \( l_1 \geq 1, l_1 + l_2 = 2^n - 2 \). Then there are two vertex-disjoint paths of lengths \( l_1, l_2 \) in \( Q_n \) with end-vertices \( u \) and \( v \).

A similar fact can be proved also in the case of even \( q(u, v) \):

2.5. Proposition. Let \( u, v \in V(Q_n) \), \( u \neq v \), \( q(u, v) \equiv 0 \, (\text{mod } 2) \); let \( l_1, l_2 \) be odd integers fulfilling \( l_1, l_2 \geq 1, l_1 + l_2 = 2^n - 2 \). Then there are two vertex-disjoint paths of lengths \( l_1, l_2 \) in \( Q_n \) with end-vertices \( u \) and \( v \).

Proof. Assume first \( l_1 \geq q(u, v) - 1, l_2 \geq q(u, v) - 1 \). It follows from 2.1 that there is a hamiltonian circuit \( c \) in \( Q_n \) such that the distances of \( u \) and \( v \) along \( c \) are \( l_1 + 1, l_2 + 1 \). Removing two suitably chosen edges from \( c \) we obtain the paths required. Suppose now e.g. \( l_1 < q(u, v) - 1 \). Let \( u' \in V(Q_n) \) such that \( q(u, u') = l_1 \), \( q(u, v) = q(u, u') + q(u', v) \). Then \( q(u', v) \) is odd and from 2.3 we conclude that there is a hamiltonian path \( p \) in \( Q_n \) with end-vertices \( u', v \) going “in the shortest possible way” from \( u' \) to \( \bar{v} \). If necessary, we can achieve by a permutation of coordinates (more exactly, by constructing a new hamiltonian path arising from \( p \)) that \( p \) goes in the shortest way from \( u' \) to \( u \). By removing one edge from \( p \) (namely that incident with \( u \) from the part of \( p \) joining \( u \) with \( v \)) we obtain the paths required.

2.6. Remark. The assumption of 2.5 that \( l_1, l_2 \) are odd cannot be omitted. (This
may be seen from the example of \( u, v \in V(Q_3) \) such that \( \varrho(u, v) = 2 \). There is no pair of vertex-disjoint paths of lengths 2 and 4 with end-vertices \( u \) and \( v \) in \( Q_3 \).

We shall need one more technical result:

2.7. Proposition. Let \( u, v, u', v' \in V(Q_n) \) be a quadruple of different vertices, let \( \varrho(u, u') = \varrho(v, v') = 1 \), \( \varrho(u, v) = \varrho(u', v') \). Then there is a pair of vertex-disjoint paths \( p_1, p_2 \) in \( Q_n \) such that \( p_1 \) joins \( u \) with \( u' \), \( p_2 \) joins \( v \) with \( v' \) and both \( p_1 \) and \( p_2 \) have the same length \( 2^{n-1} - 1 \).

Proof. We prove that, given \( u, u', v, v' \) with the properties described above, there is \( i \) \((1 \leq i \leq n)\) such that for the \( i \)-canonical decomposition of \( Q_n \) into \( Q_{n-1}' \) and \( Q_{n-1}'' \) the following holds: \( u, u' \in V(Q_{n-1}'), v, v' \in V(Q_{n-1}'') \). Then the assertion to be proved follows easily from 2.3.

Let \( u = (u_1, \ldots, u_n) \), \( v = (v_1, \ldots, v_n) \), \( u' = (u'_1, \ldots, u'_n) \), \( v' = (v'_1, \ldots, v'_n) \). Put \( J = \{ j; u_j = v_j \} \), let \( u_k \neq u'_k \), \( v_i \neq v'_i \). From the assumptions we easily derive the following assertion: if \( k = i \), then \( J - \{ k \} \neq \emptyset \); if \( k \neq i \), then \( J - \{ k, i \} \neq \emptyset \) as well. Thus it is in both cases possible to choose an integer \( i \) such that \( u_i = u'_i \neq v'_i = v_i \), q.e.d.

3. SPANNING TREES OF HYPERCUBES OBTAINED BY TRANSFORMATIONS OF BINARY TREES

In this section we describe certain spanning trees of hypercubes arising by simple transformations of binary trees.

For \( n \geq 2 \) let \( B_n \) denote a complete binary tree on \( n \) vertex-levels with one edge added to its root. Fig. 3.1 shows \( B_2, B_3 \) and \( B_4 \).

\[
\text{Fig. 3.1.}
\]

Obviously, \( B_n \) has \( 2^n - 1 + 1 \) leaves and \( 2^{n-1} - 1 \) vertices of degree 3, \( |V(B_n)| = 2^n \).

Further, denote \( B_n \) by \( B_n^{(1)} \) and for \( k \geq 2 \) let \( B_n^{(k)} \) arise from \( B_n \) by splitting each vertex into \( k \) new vertices (see examples in Fig. 3.2). Then obviously \( |V(B_n^{(k)})| = k \cdot 2^n \).

For \( k \geq 2 \) there is a unique path of length \( 2k - 1 \) joining a leaf with a vertex of degree 3 in \( B_n^{(k)} \). We call this path the main branch of \( B_n^{(k)} \) (and draw it vertically).
3.1. Remark. It follows from [3] that dim $B_n = n + 1$ for $n \geq 2$. Now we will prove that dim $B_n^{(2)} = n + 1$ as well.

![Fig. 3.2.](image)

3.2. Proposition. For $n \geq 2$, dim $B_n^{(2)} = n + 1$ and since $|V(B_n^{(2)})| = 2^{n+1}$, $B_n^{(2)}$ is a spanning tree of $Q_{n+1}$.

Proof. dim $B_n^{(2)} \geq n + 1$ follows from $|V(B_n^{(2)})| = 2^{n+1}$. In order to prove dim $B_n^{(2)} \leq n + 1$ we construct by induction a $C_{n+1}$-valuation of $B_n^{(2)}$:

a) $C_3$-valuation of $B_2^{(2)}$ is shown in Fig. 3.3.

![Fig. 3.3.](image)

b) Assume such a $C_{n+1}$-valuation $\phi$ of $B_n^{(2)}$ to be given that the two upper edges of the main branch of $B_n^{(2)}$ have values $n$ and $n + 1$ (top-down, cf. Fig. 3.4). Denote this $C_{n+1}$-valued tree $B_n^{(2)}$ by $T$. Take another copy of $B_n^{(2)}$ and construct its $C_{n+1}$-valuation $\phi'$ from $\phi$ by interchanging the values $n$ and $n + 1$, i.e. define $\phi' : E(B_n^{(2)}) \to \{1, \ldots, n + 1\}$ by putting
Denote the $C_{n+1}$-valued tree $B_n^{(2)}$ obtained in this way by $T$ (cf. Fig. 3.4).

Now, we construct a $C_{w+2}$-valued tree $B^*_n$ from $T$ and $T'$ as follows (cf. Fig. 3.5): delete two upper edges of the main branch of $T'$, join the leaf obtained by a new edge to the second from above vertex of the main branch of $T$, assign $n + 2$ to this new edge and finally add the path of length 2 (with values $n + 1$ and $n + 2$ on its edges) to the upper-most leaf (in Fig. 3.5 the new edges are drawn by thick lines).

It may be easily verified that the valuation of $B^*_n$ constructed as described is a $C_{w+2}$-valuation, q.e.d.

**3.3. Proposition.** For $n \geq 2$ and $s \geq 1$, $\dim B_n^{(2)} = n + s$ and since $|V(B_n^{(2s)})| = 2^{n+s}$, $B_n^{(2s)}$ is a spanning tree of $Q_{n+s}$.
Proof. The case $s = 1$ is solved by 3.2, assume therefore $s \geq 1$ and let a $C_{n+s}$-valuation $\varphi$ of $B_n(2^s)$ be given. Note that $B_n(2s+1)$ arises from $B_n(2^s)$ by replacing each vertex by an edge; call these edges "new" and define a valuation $\varphi': E(B_n(2s+1)) \to \{1, \ldots, n + s + 1\}$ by putting $\varphi'(e) = \varphi(e)$ if $e$ is not a new edge and $\varphi'(e) = n + s + 1$ for all the new edges. Obviously, $\varphi'$ is a $C_{n+s+1}$-valuation of $B_n(2s+1)$.

Fig. 3.6.

Fig. 3.7.
A lower bound \( \dim B_n(2^s) \geq n + s \) is trivial, since \( |V(B_n(2^s))| = 2^{n+s} \). Fig. 3.6 illustrates the construction for the case \( n = 2, s = 1 \), the new edges being again drawn by thick lines.

3.4. Corollary. For \( n \geq 2 \) and \( k > 1 \), \( \dim B_n(k) = n + \lceil \log_2 k \rceil \).

Proof. The case \( k = 2^s \) for some \( s > 0 \) is solved by 3.3. For \( k \) different from the powers of 2 we first prove the lower bound \( \dim B_n(k) \geq n + \lceil \log_2 k \rceil \), which immediately follows by comparing the cardinalities of the vertex sets \( (B_n(2^s)) \supseteq Q_m \) implies the inequality \( k \cdot 2^n \leq 2^m \). For the proof of \( \dim B_n(k) \leq n + \lceil \log_2 k \rceil \) let \( s = \lceil \log_2 k \rceil \) and let \( \varphi \) be a \( C_{n+s} \)-valuation of \( B_n(2^s) \) constructed according to 3.3. Investigating the last step of this proof, i.e., the construction of the \( C_{n+s} \)-valued \( B_n(2^s) \) from the \( C_{n+s-1} \)-valued \( B_n(2^{s-1}) \), we can see that each path of length \( 2^{s-1} \) of \( B_n(2^{s-1}) \) between two vertices of degree 3 or between a vertex of degree 3 and a leaf was extended by adding \( 2^s \) new edges to the path of length \( 2^{s+1} \) (the main branch of \( B_n(2^{s-1}) \) being extended by adding \( 2^s \) new edges to the path of length \( 2^{s+1} - 1 \)). The desired \( C_{n+s} \)-valued tree \( B_n(k) \) can be then obtained by removing arbitrarily chosen \( 2^s - k \) new edges from every such path (and by removing arbitrarily chosen \( 2(2^s - k) \) new edges from the new main branch). As an example, Fig. 3.7 shows the construction of the \( C_5 \)-valued \( B_2(5) \) from \( C_4 \)-valued \( B_2(4) \) via the \( C_5 \)-valued \( B_2(8) \).

4. FURTHER SPANNING TREES OF HYPERCUBES

4.1. Definition. For \( n \geq 3 \), any graph homeomorphic to an \( n \)-star will be called an \( n \)-quasistar.

The paths joining the centre of an \( n \)-quasistar with its leaves will be called rays of a quasistar and will be denoted by \( R_1, \ldots, R_n \). A ray is even (odd), if its length — i.e. the number of edges in it — is even (odd).

Fig. 4.1 shows two different 3-quasistars.

4.2. Remark. A bipartite graph is called balanced, if it may be regularly coloured by colours \( c_1, c_2 \) in such a way that the number of vertices coloured by \( c_1 \) equals ...
that of vertices coloured by $c_2$. Obviously, $Q_n$ is balanced ($n \geq 1$) and if $T$ is a spanning tree of $Q_n$, then $|V(T)| = 2^n$, $\text{maxdeg}(T) \leq n$ and $T$ is balanced as well. Further, an $n$-quasistar is balanced if and only if it has just one odd ray.

4.3. Proposition. Let $S$ be a balanced 3-quasistar with $|V(S)| = 2^n$ for some $n \geq 3$. Then $S$ is a spanning tree of $Q_n$ and there is such an embedding of $S$ into $Q_n$ that the images of the end-vertices of the two even rays of $S$ have distance 2 in $Q_n$.

Proof. The proof proceeds by induction on $n$. A 3-quasistar $S$ is uniquely determined by the triple $(r_1, r_2, r_3)$ of positive integers $r_1 \leq r_2 \leq r_3$, denoting the lengths of its rays. There are exactly two balanced 3-quasistars having 8 vertices; these are $(1, 2, 4)$ and $(2, 2, 3)$ and both of them are embeddable in $Q_3$ (and therefore also its spanning trees). Their embeddings satisfying the condition on the end-vertices of the even rays are shown in Fig. 4.2.

Assume now $n > 3$, let $S = (r_1, r_2, r_3)$ be a balanced 3-quasistar with rays $R_1, R_2, R_3$, let $|V(S)| = 2^n$ and hence $r_1 + r_2 + r_3 = 2^n - 1$. Recall that $r_1 \leq r_2 \leq r_3$.

1. Suppose $r_3 = 2^{n-1}$, let e.g. $r_1$ be even (in the case of $r_1$ odd and $r_2$ even we proceed quite similarly). From 2.2 we conclude that it is possible to construct a $C_{n-1}$-valuation of the path of length $2^{n-1} - 1$ formed by $R_1$ and $R_2$ so that $O(R_1) = \{1, 2\}$. We shall now extend this valuation to a $C_n$-valuation of the whole $S$ as follows: the edge of $R_3$ incident with the centre of $S$ obtains the value $n$ and the remaining part of $R_3$ which is a path of length $2^{n-1} - 1$ will be $C_{n-1}$-valued (using e.g. the basic
1. Suppose now \( r_3 > 2^{n-1} \), let e.g. \( r_1 \) be even. It follows by induction that there is a \( C_{n-1} \)-valuation of a 3-quasistar \((r_1, r_2, r_3 - 2^{n-1})\) with rays \( R_1, R_2 \) and \( R'_3 \) (where \( R'_3 \) arises from \( R_3 \) by removing the path of length \( 2^{n-1} \)). Moreover, if \( r_3 \) is odd, then \( O(R_1 + R_2) = \{1, 2\} \); if \( r_3 \) is even, then \( O(R_1 + R_3) = \{1, 2\} \). In both cases we extend this valuation by assigning values to edges of \( R_3 - R'_3 \) as follows: the edge nearest to the centre of \( S \) obtains \( n \), the remaining path \( p \) of length \( 2^{n-1} - 1 \) will be \( C_{n-1} \)-valued (using again e.g. the basic \( C_n \)-valuation) in such a way that \( O(p) = \{1\} \). Obviously we obtain a \( C_n \)-valuation of \( S \); if \( r_3 \) is odd, \( O(R_1 + R_2) = \{1, 2\} \); if \( r_3 \) is even, \( O(R_1 + R_3) = \{2, n\} \), q.e.d. (cf. Fig. 4.3).

2. Suppose now \( r_3 > 2^{n-1} \), let e.g. \( r_1 \) be even. It follows by induction that there is a \( C_{n-1} \)-valuation of a 3-quasistar \((r_1, r_2, r_3 - 2^{n-1})\) with rays \( R_1, R_2 \) and \( R'_3 \) (where \( R'_3 \) arises from \( R_3 \) by removing the path of length \( 2^{n-1} \)). Moreover, if \( r_3 \) is odd, then \( O(R_1 + R_2) = \{1, 2\} \); if \( r_3 \) is even, then \( O(R_1 + R_3) = \{1, 2\} \). In both cases we extend this valuation by assigning values to edges of \( R_3 - R'_3 \) as follows: the edge nearest to the centre of \( S \) obtains \( n \), the remaining path \( p \) of length \( 2^{n-1} - 1 \) will be \( C_{n-1} \)-valued (using again e.g. the basic \( C_n \)-valuation) in such a way that \( O(p) = \{1\} \). Obviously we obtain a \( C_n \)-valuation of \( S \); if \( r_3 \) is odd, \( O(R_1 + R_2) = \{1, 2\} \); if \( r_3 \) is even, \( O(R_1 + R_3) = \{2, n\} \), q.e.d. (cf. Fig. 4.4.).

![Fig. 4.4.](image)

3. Suppose \( r_2 > 2^{n-2} \) (and therefore also \( r_3 > 2^{n-2} \)). We remove the paths of length \( 2^{n-2} \) from \( R_2 \) and \( R_3 \) and obtain in this way a 3-quasistar \( S' = (r_1, r_2 - 2^{n-2}, r_3 - 2^{n-2}) \) with rays \( R_1, R'_2, R'_3 \). It follows by induction that there is a \( C_{n-1} \)-valuation of \( S' \) satisfying the additional condition concerning the end-vertices of the even rays. Again we will extend this \( C_{n-1} \)-valuation to the \( C_n \)-valuation of the whole \( S \); we proceed as follows:

3a. If \( r_2 \) and \( r_3 \) are even, then \( |O(R'_2 + R'_3)| = 2 \). Consider a canonical decomposition of \( Q_n \) into \( Q'_{n-1} \) and \( Q''_{n-1} \); by induction there is an embedding of \( S' \) in \( Q'_{n-1} \) such that the images \( u' \) and \( v' \) of the end-vertices of \( R'_2 \) and \( R'_3 \) have distance 2 in \( Q''_{n-1} \). We assign value \( n \) to the first edges (nearest to the centre of \( S \)) removed from \( R_2 \) and \( R_3 \). This means (in terms of the embedding) a transition from \( Q'_{n-1} \) into \( Q''_{n-1} \). The vertices \( u, v \) obtained in this way have again distance 2; choose vertices \( u'' \) and \( v'' \) in \( Q''_{n-1} \) such that \( q(u, u'') = q(v, v'') = 1 \), \( q(u'', v'') = q(u, v) = 2 \). According to 2.7 there are two vertex-disjoint paths \( p_1, p_2 \) in \( Q''_{n-1} \) such that \( p_1 \) joins \( u \) with \( u'' \), \( p_2 \) joins \( v \) with \( v'' \) and both \( p_1 \) and \( p_2 \) have length \( 2^{n-2} - 1 \). Hence we can use \( p_1 \) and \( p_2 \) for
embedding the parts of $R_2$ and $R_3$ removed from them at the beginning, q.e.d. (cf. Fig. 4.5).

3b. Let $r_1$ be even (and therefore $r_2 \neq r_3 \pmod{2}$). Without loss of generality let $r_2$ be even. Again, there is an embedding of $S'$ in $Q'_{n-1}$ such that $O(R_1 + R_2') = \{1, 2\}$; the first edges removed from $R_2$ and $R_3$ will be assigned $n$. The vertices $u$ and $v$ obtained in this way in $Q''_{n-1}$ of the canonical decomposition of $Q_n$ have an
odd distance; in order to extend the existing valuation to the whole $S$ we use again 2.7 in such a way that $|O(R_1 + R_2)| = 2$ (cf. Fig. 4.6).

4. Suppose that neither the case 1 nor 2 nor 3 holds. Then necessarily $r_1 = r_2 = 2^{n-2}$ and $r_3 = 2^{n-1} - 1$. To see it recall that $r_1 \leq r_2 \leq r_3$ and $r_1 + r_2 + r_3 = 2^n - 1$. Hence $r_1 + r_2 < 2^{n-1}$ implies $r_3 \geq 2^{n-1}$ and the case 1 or 2 would follow; therefore $r_1 + r_2 \geq 2^{n-1}$ and since $r_1 \leq r_2$ and $r_2 \leq 2^{n-2}$ (otherwise the case 3 would take place), the desired equalities follow.

In order to construct a $C_n$-valuation of $S$ we proceed in this case as follows (cf. Fig. 4.7): assign to edges of $R_1$ (which is of length $2^{n-2}$) from the end to the centre of $S$ the values of the basic $C_{n-2}$-valuation; to the edge incident with the centre give the value $n - 1$; to edges of $R_2$ (which is of length $2^{n-2}$ as well) we assign (from the centre to the end) again the values of the basic $C_{n-2}$-valuation, while the edge incident with the leaf obtains $n$. The edges of $R_3$ (of length $2^{n-1} - 1$) are treated in the following way: the edge incident with the centre obtains $n$, the others (in the direction to the leaf) the values of the basic $C_{n-1}$-valuation, the last value ($= 1$) not being used, since the length of the whole $R_3$ is only $2^{n-1} - 1$. Thereafter we interchange the values $n - 1$ and $n - 2$. It may be easily checked that the valuation of $S$ obtained is its $C_n$-valuation fulfilling $O(R_1) = \{n - 2, n - 1\}$, $O(R_2) = \{n - 2, n\}$, therefore $|O(R_1 + R_2)| = 2$, which completes the proof of the whole proposition.

The following statement describes another class of spanning trees of hypercubes.

4.4. Proposition. Let $T$ be a tree fulfilling the following conditions: $T$ is balanced, $|V(T)| = 2^n$ for some $n \geq 3$, maxdeg $(T) = 3$ and $T$ has exactly 2 vertices of degree 3. Then $T$ is a spanning tree of $Q_n$. 
Proof. A tree $T$ fulfilling the assumptions is uniquely determined by the 5-tuple of positive integers $(r_1, r_2, a, r_3, r_4)$, where $r_1, \ldots, r_4$ are the lengths of the four rays $R_1, \ldots, R_4$ of $T$ and $a$ is the length of the axial path $A$ of $T$ (cf. Fig. 4.8).

![Fig. 4.8.]

We have $a + r_1 + r_2 + r_3 + r_4 = 2^n - 1$, assume without loss of generality $r_1 + r_2 \leq r_3 + r_4$.

1. Let $r_3 + r_4 > 2^n - 1$; suppose e.g. $r_3 \leq r_4$. If $r_3 > 2$, then it is possible to remove an even positive number of edges both from $R_3$ and $R_4$ in such a manner that altogether $2^n - 1$ edges are deleted; the tree $(r_1, r_2, a, r_3', r_4')$ obtained in this way has $2^n - 1$ edges and obviously is balanced. Therefore (by induction), it is a spanning tree of $Q_{n-1}$; let us assign $n$ to the first edges of the removed parts of $R_3$ and $R_4$ (in terms of the embedding this means a transition from $Q_{n-1}'$ to $Q_{n-1}''$ in the canonical decomposition of $Q_n$ into $Q_{n-1}'$ and $Q_{n-1}''$). Since the remaining parts of $R_3$ and $R_4$ have odd lengths it is possible to extend the construction of the $C_n$-valuation to the whole $T$ according to 2.4 or 2.5 (cf. Fig. 4.9).

We proceed similarly also in the case $r_3 = 2$ — then we remove the whole $R_3$ and obtain a balanced 3-quasistar $(r_1, r_2, a + r_4 - 2^{n-1} + 2)$ (cf. Fig. 4.10).

Let now $r_3 = 1$, then $r_4 \leq 2^{n-1}$. We delete $2^{n-1}$ edges from $R_4$ and obtain either a quasistar or a tree $(r_1, r_2, a, 1, r_4 - 2^{n-1})$; both of them are spanning trees of $Q_{n-1}$.

![Fig. 4.9.]

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Let us extend the corresponding $C_{n-1}$-valuation as follows: the first edge obtains $n$ (as usual it means a transition to $Q_{n-1}$ in a canonical decomposition) and for the remaining path of length $2^{n-1} - 1$ we can use e.g. the basic $C_{n-1}$-valuation, q.e.d.

Fig. 4.10.

2. Let $r_3 + r_4 < 2^{n-1}$ (and therefore $r_1 + r_2 < 2^{n-1}$ as well). Then there is an edge of the axial path such that by removing it we obtain from $T$ two graphs having $2^{n-1} - 1$ edges which are either balanced 3-quasistars or paths. Hence, by induction, they are spanning trees of $Q_{n-1}$ and it suffices to take their corresponding $C_{n-1}$-valuations and to assign $n$ to the edge that has been previously removed.

3. Let $r_3 + r_4 = 2^{n-1}$, both $r_3$ and $r_4$ being even. We delete from $T$ the whole rays $R_3$ and $R_4$ and obtain in this way a balanced 3-quasistar $S$ which is a spanning tree of $Q_{n-1}$. Let $u$ be a vertex of $T$ incident with $R_3$, $R_4$ and $A$ (then $u$ is obviously a leaf of $S$ — cf. Fig. 4.11). We extend an existing $C_{n-1}$-valuation of $S$ to the whole $T$ as follows: change the value $i$ of the (only) edge of $S$ incident with $u$ to $n$ and assign $n$ also to the last edge of $R_4$. Let $R_4'$ denote the rest of $R_4$ after removing the last edge; it is possible (using 2.2) to construct a $C_{n-1}$-valuation of $R_3 + R_4'$ (whose length is $2^{n-1} - 1$) such that $O(R_4') = \{i\}$. Then it may be easily checked that in this way a $C_n$-valuation of $T$ arises, q.e.d.

Fig. 4.11.
4. Let \( r_3 + r_4 = 2^{n-1} \), both \( r_3 \) and \( r_4 \) being odd. Since \( T \) is balanced, \( r_1, r_2 \) and \( a \) have to be odd as well. We proceed as follows (cf. Fig. 4.12): let \( u \) be the vertex of \( A \) whose distance from the common vertex of \( R_3 \) and \( R_4 \) equals 1 (if \( a = 1 \), then \( u \) is incident with both \( R_1 \) and \( R_2 \)); let us remove from \( T \) the whole \( R_3 \) and \( R_4 \) and also the last edge of \( A \) (incident with \( u \)). Denote the graph obtained by \( S \). Obviously \( S \) has \( 2^{n-1} - 2 \) edges; if \( a > 1 \), then \( S \) is a non-balanced 3-quasistar \((r_1, r_2, a - 1)\), if \( a = 1 \), then \( S \) is a path. Add for a moment a new edge to \( R_1 \) in \( S \), let \( S' \) be the graph obtained and let \( v \) be the new end-vertex of the extended \( R_1 \) in \( S' \). Denote by \( p \) the path in \( S' \) joining \( u \) and \( v \). We shall construct a \( C_{n-1} \)-valuation of \( S' \) such that \( |O(p)| = 2 \); use for it 3.3 in case that \( a > 1 \) and therefore \( S' = (r_1 + 1, r_2, a - 1) \) is a balanced 3-quasistar, and 2.2 if \( a = 1 \) and \( S' \) is a path. This \( C_{n-1} \)-valuation partialized to \( S \) will be the starting point of the construction of the desired \( C_n \)-valuation of \( T \): let us assign \( n \) to the last edge of \( A \) (having been previously removed) and also to the last edge of \( R_4 \); further, we construct a \( C_{n-1} \)-valuation of the remaining part \( R_4' \) of \( R_4 \) and of the whole \( R_3 \) in such a way that \( O(R_4') = O(p) \). It may be easily checked that we have obtained a \( C_n \)-valuation of \( T \), q.e.d. This completes the proof of the whole proposition.

5. CONCLUDING REMARKS, OPEN PROBLEMS AND CONJECTURES

The propositions proved in the previous sections might be useful when trying to solve the following

5.1. **Open problem.** Characterize the spanning trees of \( Q_n \).

Let us note here that the conditions mentioned in 4.2, necessary for \( T \) to be a spanning tree of \( Q_n \) (namely, that \( |V(T)| = 2^n \), \( T \) is balanced and \( \maxdeg(T) \leq n \)) are not sufficient. In order to see this start from the so called 4-tomic tree on 2 levels of edges, denoted by \( T_2^{(4)} \) (cf. Fig. 5.1). It is proved in [4] that for \( k \geq 2 \), \( \dim T_2^{(k)} = 151 \).
= \lceil (3k + 1)/2 \rceil$, hence $\dim T_2^{(4)} = 7$. We can easily construct (by adding new vertices and edges to $T_2^{(4)}$) a balanced tree $T'$ with 64 vertices and $\maxdeg(T') = 5$ such that $\dim T' \geq 7$; hence, $T'$ cannot be a spanning tree of $Q_6$.

On the other hand, it seems not quite hopeless to try to strengthen further 4.4 (cf. also 4.3), possibly to the following

5.2. **Conjecture.** Let $T$ be a balanced tree, $|V(T)| = 2^n$, $\maxdeg(T) \leq 3$. Then $T$ is a spanning tree of $Q_n$.

We recall in this connection that [8] contains two examples of spanning trees of $Q_n$ with maximal degree 3 having a large number of vertices of degree 3.

Another way of generalizing 4.3 is the following

5.3. **Conjecture.** Let $T$ be a balanced $l$-quasistar, $|V(T)| = 2^n$, $l \leq n$. Then $T$ is a spanning tree of $Q_n$.

[9] contains the proof of the latter conjecture for $l = 4$ and 5.

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**References**


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