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On quasistars in \( n \)-cubes

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If \( m \geq 3 \) is an integer, then a graph (in the sense of [1]) which is homeomorphic to the star \( K(1, m) \) will be referred to as an \( m \)-quasistar. Let \( T \) be an \( m \)-quasistar \((m \geq 3)\) of order \( p \); obviously, \( T \) is a tree and \( p \geq m + 1 \); we say that \( T \) is balanced if \( p \) is even and there exists a 2-coloring of \( T \) with \( p/2 \) red vertices and \( p/2 \) green ones.

The present note was inspired by I. Havel's paper [2]. Let \( m \) and \( n \) be integers, \( 3 \leq m \leq n \), and let \( T \) be a balanced \( m \)-quasistar of order \( 2^n \). Havel conjectured that \( T \) can be embedded into the \( n \)-cube; he proved this conjecture for the case when \( m = 3 \). In the present note we shall prove this conjecture for the cases when \( m = 4 \) and \( 5 \). Moreover, we shall give an alternative proof of the case \( m = 3 \).

Let \( G \) be an \( n \)-cube, \( n \geq 1 \). Then there exist vertex-disjoint \((n - 1)\)-cubes \( G' \) and \( G'' \) such that \( V(G) = V(G') \cup V(G'') \); we shall say that \( G \) can be partitioned into \( G' \) and \( G'' \). Let \( u' \in V(G') \); the only vertex \( u'' \in V(G'') \) with the property that \( (u', u'') \in E(G) \) will be denoted by \( u''(G'') \).

Let \( P \) be a nontrivial path. Then \( P \) is homeomorphic to \( K_2 \). If \( u \) is a vertex of degree one in \( P \), then \( P \) will be referred to as a \( u \)-path. Assume that \( P \) is a \( u \)-path. Then the only vertex of degree one in \( P \) which is different from \( u \) will be denoted by \( e(P, u) \).

**Lemma 1.** Let \( G \) be an \( n \)-cube, \( n \geq 3 \), and let \( u_1, u_2, \bar{u}_1, \bar{u}_2 \in V(G) \), \( u_1 \neq u_2 \). Assume that \( a_1 \) and \( a_2 \) are even positive integers such that \( a_1 + a_2 = 2^n \). Then there exist vertex-disjoint paths \( P_1 \) and \( P_2 \) in \( G \) with the property that for \( i \in \{1, 2\} \), \( P_i \) is a \( u_i \)-path of order \( a_i \) such that \( e(P_i, u_i) \neq \bar{u}_i \).

**Proof.** The case of \( n = 3 \) is easy. Let \( n = n_0 \geq 4 \); assume that for \( n = n_0 - 1 \), the lemma was proved. Clearly, \( G \) can be partitioned into two vertex-disjoint \((n - 1)\)-cubes \( G_1 \) and \( G_2 \) in such a way that \( u_1 \in V(G_1) \) and \( u_2 \in V(G_2) \). Without loss of generality we shall assume that \( a_1 \geq a_2 \). If \( a_1 = a_2 \), then there exists a Hamiltonian \( u_i \)-path in \( G_i \) such that \( \bar{u}_i \neq e(P_i, u_i) \) for \( i = 1, 2 \), and thus the lemma is proved.

We shall assume that \( a_1 > a_2 \). Then there exists a Hamiltonian \( u_1 \)-path \( P' \) in \( G_1 \) such that \( u_2(G_1) = e(P', u_1) \). Denote \( w = e(P', u_1) \). It follows from the induction assumption that there exist vertex-disjoint paths \( P'' \) and \( P_2 \) in \( G_2 \) with the properties...
that $P''$ is a $w(G_2)$-path of order $a_1 - 2^{n-1}t\bar{u}_1 + \varepsilon(P'', w(G_2))$, $P_2$ is a $u_2$-path of order $a_2$, and $\bar{u}_2 + \varepsilon(P_2, u_2)$. We denote by $P_1'$ the path induced by the edges $E(P') \cup \{w(G_2)\} \cup E(P'')$. It is clear that the paths $P_1$ and $P_2$ have the desired properties.

**Lemma 2.** Let $k \in \{1, 2, 3\}$, let $G$ be an $n$-cube, where $n \geq k$, let $u_1, \ldots, u_k$ be $k$ distinct vertices of $G$, and let $a_1, \ldots, a_k$ be even positive integers such that $a_1 + \ldots + a_k = 2^n$. Then there exist vertex-disjoint paths $P_1, \ldots, P_k$ in $G$ such that $P_i$ is an $u_i$-path of order $a_i$ for each $i \in \{1, \ldots, k\}$.

**Proof.** The case of $k = 1$ is obvious. The case of $k = 2$ is obvious for $n = 2$, and follows immediately from Lemma 1 for $n \geq 3$. Let $k = 3$. The proof of the lemma is very easy for $n = 3$. Assume that $n \geq 4$. It is clear that $G$ contains four vertex-disjoint $(n - 2)$-cubes $G_1, G_2, G_3, G_4$ such that $u_i \in V(G_i)$ for $i = 1, 2, 3$. Without loss of generality we may assume that $V(G_1) \cup V(G_2)$ induces an $(n - 1)$-cube in $G$, and that $V(G_4) \cup V(G_1)$ also induces an $(n - 1)$-cube in $G$. If $a_1 + a_2 \leq 2^{n-1}$ and $a_2 + a_3 \leq 2^{n-1}$, then the fact that $a_1 + a_2 + a_3 = 2^n$ implies that $a_2 \leq 0$, which is a contradiction. Thus, without loss of generality we shall assume that $a_1 + a_2 > 2^{n-1}$. We denote by $G'$ or $G''$ the $(n - 1)$-cube in $G$ which is induced by $V(G_1) \cup \cup V(G_2)$ or by $V(G_3) \cup V(G_4)$, respectively. There exists a permutation $\pi$ on $\{1, 2\}$ such that $a_{\pi(1)} \geq a_{\pi(2)}$. It is clear that $a_{\pi(2)} \leq 2^{n-1} - 2$. It follows from Lemma 1 that there exist vertex-disjoint paths $P'$ and $P_{\pi(2)}$ in $G'$ such that $P'$ is a $u_{\pi(1)}$-path of order $2^{n-1} - a_{\pi(2)}$, $u_3(G') + \varepsilon(P', u_{\pi(1)})$, and $P_{\pi(2)}$ is a $u_{\pi(2)}$-path of order $a_{\pi(2)}$. Denote $w = \varepsilon(P', u_{\pi(1)})$. It follows from the case $k = 2$ of the present lemma that there exist vertex-disjoint paths $P''$ and $P_3$ in $G''$ such that $P''$ is a $w(G'')$-path of order $a_{\pi(1)} + a_{\pi(2)} - 2^{n-1}$, and $P_3$ is a $u_3$-path of order $a_3$. We denote by $P_{\pi(1)}$ the path induced by the edges $E(P') \cup \{w(G'')\} \cup E(P'')$. It is clear that the paths $P_1, P_2, P_3$ have the desired properties, which completes the proof.

Let $m$ and $n$ be integers such that $2 \leq m \leq n$. We denote by $R(m, n)$ the set of sequences $(r_1, \ldots, r_m)$ of positive integers with the properties that $r_1 + \ldots + r_m = 2^n - 1$ and that $r_i$ is odd for exactly one $i \in \{1, \ldots, m\}$.

**Lemma 3.** Let $n \geq 3$, and let $(r_1, r_2, r_3) \in R(3, n)$. Then there exist an even integer $s \geq 0$ and a permutation $\pi$ on $\{1, 2, 3\}$ such that $r_{\pi(1)}$ is even and $(r_{\pi(2)} - s, r_{\pi(3)}) \in R(2, n - 1)$.

**Proof.** Without loss of generality we assume that $r_1 \geq r_2 \geq r_3$. If $r_1 + r_2 \leq 2^{n-1}$, then $2^n - 1 = r_1 + r_2 + r_3 \leq 2^{n-1} + 2^{n-2}$ and therefore $n \leq 2$, which is a contradiction. We shall assume that $r_1 + r_2 \geq 2^{n-1} + 1$.

Let first $r_1 \geq 2^{n-1} + 1$. Then there exists a permutation $\pi$ on $\{1, 2, 3\}$ such that $\pi(2) = 1$ and $r_{\pi(1)}$ is even. It is obvious that $(r_{\pi(2)} - (2^{n-1} - r_{\pi(1)}), r_{\pi(3)})$ belongs to $R(2, n - 1)$.

Let now $r_1 \leq 2^{n-1}$. There exists a permutation $\pi$ on $\{1, 2, 3\}$ such that $\pi(3) = 3$ and $r_{\pi(1)}$ is even. It is obvious that $(r_{\pi(2)} - (2^{n-1} - r_{\pi(1)}), r_{\pi(3)})$ belongs to $R(2, n - 1)$.  

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Lemma 4. Let $m \in \{4, 5\}$, let $n \geq m$, and let $(r_1, \ldots, r_m) \in R(m, n)$. Assume that $r_1 \geq \ldots \geq r_m$. Then there exist even integers $s \geq 0$ and $t \geq 0$, and a permutation $\pi$ on $\{1, 2, 3\}$ such that $r_{\pi(1)}$ is even, and

$$(r_{\pi(2)} - s, r_{\pi(3)} - t, r_4, \ldots, r_m) \in R(m - 1, n - 1).$$

Proof. If $r_1 + r_2 + r_3 \geq 2^{n-1} + 3$, then the statement of the lemma follows easily.

Let $r_1 + r_2 + r_3 \leq 2^{n-1} + 2$. Then

$$2^n - 1 = r_1 + \ldots + r_m \leq m(2^{n-1} + 2)/3.$$ 

This implies that $(6 - m)2^{n-1} < 2m + 3$. Since $n \geq m$, we get that $m \notin \{4, 5\}$, which is a contradiction. Thus the lemma is proved.

Theorem. Let $m \in \{3, 4, 5\}$ and let $n$ be an integer such that $n \geq m$. Then every balanced $m$-quasistar of order $2^n$ can be embedded into the $n$-cube.

Proof. Let $T$ be a balanced $m$-quasistar of order $2^n$, and let $G$ be an $n$-cube. Clearly, $G$ can be partitioned into two vertex-disjoint $(n - 1)$-cubes, say $G'$ and $G''$.

Obviously, $n \geq 3$. If $n > 3$, assume that the theorem holds for $n - 1$. We denote by $w$ the vertex of degree $m$ in $T$. Let $w_1, \ldots, w_m$ be distinct vertices of degree one in $T$. We denote by $r_i$ the distance between $w$ and $w_i$ in $T$ for $1 \leq i \leq m$. It is easy to see that $(r_1, \ldots, r_m)$ belongs to $R(m, n)$. It follows from Lemmas 3 and 4 that there exist even integers $s$ and $t$ and a permutation $\pi$ on $\{1, \ldots, m\}$ with the properties that

$s \geq t \geq 0,$

$r_{\pi(1)}$ is even,

if $m = 3$, then $t = 0$ and $(r_{\pi(2)} - s, r_{\pi(3)})$ belongs to $R(2, n - 1),$

if $m \geq 4$, then $(r_{\pi(2)} - s, r_{\pi(3)} - t, r_{\pi(4)}, \ldots, r_{\pi(m)})$ belongs to $R(m - 1, n - 1)$.

Let $k$ be the integer defined as follows: if $s = 0$, then $k = 1$; if $s > 0$ and $t = 0$, then $k = 2$; and if $t > 0$, then $k = 3$. There exist distinct vertices $u_1, v_1, \ldots, u_k, v_k$ of $T$ with the following properties:

$u_i v_i \in E(T)$ and $v_i$ belongs to the $u_i - w_{\pi(i)}$ path in $T$ for every $i \in \{1, \ldots, k\};$

$u_1 = w;$

if $k \geq 2$, then the distance between $u_2$ and $w_{\pi(2)}$ is $s;$

if $k = 3$, then the distance between $u_3$ and $w_{\pi(3)}$ is $t.$

Let $T'$ be the component of $T - u_1 v_1 - \ldots - u_k v_k$ which contains $w$. Then $T'$ is a tree of order $2^{n-1}$. If $T'$ is a path, then $T'$ can be embedded into $G'$. Assume that $T'$ is not a path. Then $m \geq 4$. Since $r_{\pi(1)}$, $s$ and $t$ are even, $T'$ is a balanced $(m - 1)$-quasistar. According to the induction assumption, $T'$ can be embedded into $G'$. Thus, we can assume that $T'$ is a subgraph of $G'$.

It follows from Lemma 2 that there exist vertex-disjoint paths $P_1, \ldots, P_k$ in $G''$ with the following properties:
$P_1$ is a $u_1(G^{r})$-path of order $r_{n(1)}$;
if $k \geq 2$, then $P_2$ is a $u_2(G^{s})$-path of order $s$; and
if $k = 3$, then $P_3$ is a $u_3(G^{t})$-path of order $t$.

The subgraph of $G$ induced by

$$E(T') \cup E(P_1) \cup \ldots \cup E(P_k) \cup \{u_1u_1(G''), \ldots, u_ku_k(G'')\}$$

is isomorphic to $T$, which completes the proof.

References


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