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GRAPHS WITH NON-ISOMORPHIC VERTEX NEIGHBOURHOODS
OF THE FIRST AND SECOND TYPES

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Summary. The paper is devoted to the relation between the classes $\mathcal{G}_1$, $\mathcal{G}_2$ of graphs with non-isomorphic vertex neighbourhoods of the first and second types; the main theorem of the paper implies that each of the classes $\mathcal{G}_1 - \mathcal{G}_2$, $\mathcal{G}_2 - \mathcal{G}_1$, $\mathcal{G}_1 \cap \mathcal{G}_2$ is infinite.

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INTRODUCTION

Let $G = (V(G), E(G))$ be a finite undirected graph without loops and multiple edges, $u \in V(G)$ its vertex. The neighbourhood of $u$ (defined in the obvious sense, i.e., as the induced subgraph on the set of all vertices which are adjacent to $u$ in $G$) will be referred to as the neighbourhood of the first type of $u$ and denoted by $N_1(u, G)$. We say that an edge $vw \in E(G)$ is adjacent to $u$ if $v \neq u \neq w$ and either $v$ or $w$ is adjacent to $u$. According to [3], [5], [2] we define the “line-version” of $N_1(u, G)$ as follows: The neighbourhood of the second type of $u$ (denoted by $N_2(u, G)$) is the edge-induced subgraph (see e.g. [1], [6]) on the set of all edges which are adjacent to $u$. (More precisely: the edge set of $N_2(u, G)$ contains all the edges $vw \in E(G)$ for which $\min \{q(v, u), q(w, u)\} = 1$, $q(x, y)$ denoting the distance of vertices $x, y$).

J. Sedláček [3], [5] introduced the following classes $\mathcal{G}_1, \mathcal{G}_2$ of asymmetrical graphs: $\mathcal{G}_1$ contains all graphs $G$ such that for every pair of distinct vertices $u, v \in V(G)$ the neighbourhoods of the $i$-th type $N_i(u, G), N_i(v, G)$ are non-isomorphic, $i = 1, 2$.

In [3] it is shown that for every integer $n \geq 6$ there exists a graph $G_n \in \mathcal{G}_1$ with $n$ vertices; the corresponding graph $G_6$ (with the minimum number of vertices) is shown in Fig. 1. The analogous question for the class $\mathcal{G}_2$ is solved in [2]: A graph $G_n \in \mathcal{G}_2$ with $n$ vertices exists if and only if $n \geq 7$; the corresponding minimal graph $G_7$ with 7 vertices is shown in Fig. 2.

As shown in [5], the graph in Fig. 1 belongs, in fact, to $\mathcal{G}_1 - \mathcal{G}_2$, and hence $\mathcal{G}_1 \neq \mathcal{G}_2$; analogously, the graph in Fig. 2 belongs to $\mathcal{G}_2 - \mathcal{G}_1$, and hence
Further, an example is given in [5] of a graph with 8 vertices which belongs to $\mathcal{G}_1 \cap \mathcal{G}_2$; hence $\mathcal{G}_1 \cap \mathcal{G}_2 \neq \emptyset$. In the present paper we shall show that each of the classes $\mathcal{G}_1 - \mathcal{G}_2$, $\mathcal{G}_2 - \mathcal{G}_1$, $\mathcal{G}_1 \cap \mathcal{G}_2$ is infinite, and we shall find the minimal member in the last of them.

**Fig. 1**

**Fig. 2**

**MAIN THEOREM**

**Theorem.** Let $n$ be an integer. Then there exists a graph $G_n$ with $n$ vertices which belongs to the class

a) $\mathcal{G}_1 - \mathcal{G}_2$ if and only if $n \geq 6$,

b) $\mathcal{G}_2 - \mathcal{G}_1$ if and only if $n \geq 7$,

c) $\mathcal{G}_1 \cap \mathcal{G}_2$ if and only if $n \geq 7$.

**Corollary.** Each of the classes $\mathcal{G}_1 - \mathcal{G}_2$, $\mathcal{G}_2 - \mathcal{G}_1$, $\mathcal{G}_1 \cap \mathcal{G}_2$ is infinite.

We shall first prove some auxiliary assertions. We say that a vertex $u \in V(G)$ is **universal** if it is adjacent to all the other vertices of $G$.

**Lemma 1.** Let $n \geq 6$ be an integer; suppose that $G_n$ is a connected graph having $n$ vertices $u_1, \ldots, u_n$, and that none of them is universal. Let us construct the graph $G_{n+1}$ with $n+1$ vertices by adding a new vertex $u_{n+1}$ to $G_n$ and making it universal in $G_{n+1}$. Then

a) $G_n \subseteq \mathcal{G}_1 \implies G_{n+1} \subseteq \mathcal{G}_1$,

b) $G_n \subseteq \mathcal{G}_2 \implies G_{n+1} \subseteq \mathcal{G}_2$.

**Proof.** Let $i = 1$ or $i = 2$ and $G_n \subseteq \mathcal{G}_i$; suppose $G_{n+1} \notin \mathcal{G}_i$, i.e., for some distinct vertices $u_a, u_b \in V(G_{n+1})$ there exists an isomorphism $f: N_i(u_a, G_{n+1}) \to N_i(u_b, G_{n+1})$. Since $u_{n+1}$ is universal in $N_i(u_j, G_{n+1})$ for $1 \leq j \leq n$ while $N_i(u_{n+1}, G_{n+1}) \simeq G_n$ has no universal vertex, necessarily $a \leq n$ and $b \leq n$ ($\simeq$ denotes isomorphism).
Hence either \( f(u_{n+1}) = u_{n+1} \) and then the partial mapping \( f|_{V(N_i(u_n, G_n))} \) is an isomorphism \( N_i(u, G_n) \) onto \( N_i(u, G_n) \), which is impossible, or \( f(u_{n+1}) \) is another universal vertex \( u_\gamma \) in \( N_i(u, G_{n+1}) \), and in this case interchanging the universal vertices \( u_\gamma, u_{n+1} \) we again obtain a contradiction.

2. If, conversely, \( G_n \notin N_i \) for \( i = 1 \) or \( i = 2 \), then we have an isomorphism \( f: N_i(u, G_n) \rightarrow N_i(u, G_n) \); defining \( f(u_{n+1}) = u_{n+1} \) we obtain an isomorphism \( f: N_i(u, G_{n+1}) \rightarrow N_i(u, G_{n+1}) \) and hence \( G_{n+1} \notin N_i \).

Lemma 2. Let \( n \geq 6 \) be an integer; suppose that \( G_n \) is a graph with \( n \) vertices \( u_1, \ldots, u_n \) such that the only universal vertex in \( G_n \) is \( u_n \) and that the minimum degree of \( G_n \) is at least 2. Let us construct the graph \( G_{n+1} \) with \( n + 1 \) vertices by adding a new vertex \( u_{n+1} \) to \( G_n \) and joining it to \( u_n \) by an edge. Then

a) \( G_n \in N_1 \leftrightarrow G_{n+1} \in N_1 \),
b) \( G_n \in N_2 \leftrightarrow G_{n+1} \in N_2 \).

Proof. a) 1. Let \( G_n \in N_1 \). Evidently \( N_1(u_i, G_n) = N_1(u_i, G_{n+1}) \) for \( 1 \leq i \leq n - 1 \); moreover, \( u_n \) is the only vertex of degree \( n \) in \( G_{n+1} \) and \( u_{n+1} \) is the only vertex of degree 1 in \( G_{n+1} \). Hence \( G_{n+1} \in N_1 \).

2. Suppose conversely that \( G_n \notin N_1 \), i.e., some distinct vertices \( u_\alpha, u_\beta \in V(G_n) \) have isomorphic neighbourhoods. Since \( u_n \) is the only universal vertex in \( G_n \), necessarily \( \alpha \neq n \neq \beta \); hence

\[ N_1(u_\alpha, G_{n+1}) = N_1(u_\alpha, G_n) \approx N_1(u_\beta, G_n) = N_1(u_\beta, G_{n+1}) \]

and therefore \( G_{n+1} \notin N_1 \).

b) 1. Let \( G_n \in N_2 \) and suppose that \( G_{n+1} \notin N_2 \), i.e., there exists an isomorphism \( f: N_2(u_\alpha, G_{n+1}) \rightarrow N_2(u_\beta, G_{n+1}) \) for some \( u_\alpha, u_\beta \in V(G_{n+1}) \), \( u_\alpha \neq u_\beta \). First observe that the neighbourhoods of \( u_i \) for \( i \neq n \) have \( n \) vertices while \( N_2(u, G_{n+1}) \) has \( n - 1 \) vertices; hence \( \alpha \neq n \neq \beta \). Further, evidently \( N_2(u_{n+1}, G_{n+1}) \approx K_{1,n-1} \). If \( \alpha = n + 1 \) then \( N_2(u_\beta, G_{n+1}) \approx K_{1,n-1} \) and \( 1 \leq \beta \leq n - 1 \); considering neighbourhoods of the neighbouring vertices of \( u_\beta \) we obtain a contradiction. Hence \( \alpha \neq n + 1 \); similarly \( \beta \neq n + 1 \) and therefore \( 1 \leq \alpha, \beta \leq n - 1 \). The vertex \( u_{n+1} \) has degree 1 both in \( N_2(u_\alpha, G_{n+1}) \) and in \( N_2(u_\beta, G_{n+1}) \); hence either \( f(u_{n+1}) = u_{n+1} \) and then the partial mapping \( f|_{V(N_2(u_\alpha, G_{n+1}))} \) is an isomorphism \( N_2(u_\alpha, G_n) \) onto \( N_2(u_\beta, G_n) \), which is impossible, or \( f(u_{n+1}) \) is another vertex \( u_\gamma \) of degree 1 in \( N_2(u_\beta, G_n) \) and in this case by interchanging the vertices \( u_{n+1}, u_\gamma \) we again obtain a contradiction.

2. Suppose conversely that \( G_n \notin N_2 \), i.e., we have an isomorphism \( f: N_2(u_\alpha, G_n) \rightarrow N_2(u_\beta, G_n) \) for some \( u_\alpha, u_\beta \in V(G_n) \), \( \alpha \neq \beta \). Necessarily \( \alpha \neq n \neq \beta \) since \( u_n \) is universal in \( N_2(u_i, G_n) \) for \( 1 \leq i \leq n - 1 \) while \( N_2(u_n, G_n) \) has no universal vertex. Further, \( u_n \) is the only vertex of degree \( n - 1 \) both in \( N_2(u_\alpha, G_n) \) and in \( N_2(u_\beta, G_n) \), and hence \( f(u_n) = u_n \). Therefore, if we define \( f(u_{n+1}) = u_{n+1} \), we obtain an isomorphism \( N_2(u_\alpha, G_{n+1}) \) onto \( N_2(u_\beta, G_{n+1}) \), i.e. \( G_{n+1} \notin N_2 \).
Lemma 3. Let \( n \geq 6 \) be an integer; suppose that \( G_n \) is a graph with \( n \) vertices \( u_1, \ldots, u_n \) such that the only universal vertex in \( G_n \) is \( u_{n-1} \) and the only vertex of degree 1 in \( G_n \) is \( u_n \). Let us construct the graph \( G_{n+1} \) with \( n + 1 \) vertices by adding a new vertex \( u_{n+1} \) to \( G_n \) and joining it to \( u_n \) by an edge. Then

a) \( G_n \in \mathcal{G}_1 \iff G_{n+1} \in \mathcal{G}_1 \),

b) \( G_n \in \mathcal{G}_2 \iff G_{n+1} \in \mathcal{G}_2 \).

Proof. a) 1. If \( G_n \in \mathcal{G}_1 \), then, since \( N_1(u_i, G_n) = N_1(u_i, G_{n+1}) \) for \( 1 \leq i \leq n - 1 \), \( N_1(u_{n+1}, G_{n+1}) \) is the graph which consists of an isolated vertex and \( N_1(u_n, G_{n+1}) \) consists of two isolated vertices, evidently \( G_{n+1} \in \mathcal{G}_1 \).

2. If, conversely, \( G_n \notin \mathcal{G}_1 \), then there exist vertices \( u_\alpha, u_\beta, \alpha \neq \beta \), such that \( N_1(u_\alpha, G_n) \simeq N_1(u_\beta, G_n) \). Evidently \( 1 \leq \alpha, \beta \leq n - 1 \) and hence \( N_1(u_\alpha, G_{n+1}) = N_1(u_\beta, G_{n+1}) \), i.e. \( G_{n+1} \notin \mathcal{G}_1 \).

b) 1. If \( G_n \in \mathcal{G}_2 \), then evidently \( G_{n+1} \in \mathcal{G}_2 \), since \( N_2(u_i, G_{n+1}) = N_2(u_i, G_n) \) for \( 1 \leq i \leq n, i \neq n - 1 \), and these neighbourhoods have \( n - 1 \) vertices and are connected, while \( N_2(u_{n-1}, G_{n+1}) \) is disconnected and \( N_2(u_{n+1}, G_{n+1}) \) has exactly two vertices.

2. If, conversely, \( G_n \notin \mathcal{G}_2 \), then \( N_2(u_\alpha, G_n) \simeq N_2(u_\beta, G_n) \) for some \( \alpha \neq \beta \). One can easily observe that necessarily \( \alpha + n - 1 \neq \beta \) and hence evidently \( G_{n+1} \notin \mathcal{G}_2 \).

Proof of the theorem. The assertion concerning the non-existence of the graph \( G_n \in \mathcal{G}_1 - \mathcal{G}_2 \) with \( n \) vertices for \( n \leq 5 \) is contained in \([3]\), the non-existence of the graph \( G_n \) on \( n \) vertices which belongs either to \( \mathcal{G}_2 - \mathcal{G}_1 \) or to \( \mathcal{G}_1 \cap \mathcal{G}_2 \) follows for \( n \leq 6 \) from \([2]\), Theorem 2.1.

a) For \( n \geq 6 \) define the graph \( G_n \in \mathcal{G}_1 - \mathcal{G}_2 \) by using the following construction:

- for \( n = 6 \) see the graph \( G_6 \) in Fig. 1;
- having obtained \( G_n \), construct \( G_{n+1} \) using
  - Lemma 1 for \( n \equiv 0 \) (mod 3),
  - Lemma 2 for \( n \equiv 1 \) (mod 3),
  - Lemma 3 for \( n \equiv 2 \) (mod 3).

b) For \( n \geq 7 \) define the graph \( G_n \in \mathcal{G}_2 - \mathcal{G}_1 \) by using the following construction:

- for \( n = 7 \) see the graph \( G_7 \) in Fig. 2;
- having obtained \( G_n \), construct \( G_{n+1} \) using
  - Lemma 1 for \( n \equiv 1 \) (mod 3),
  - Lemma 2 for \( n \equiv 2 \) (mod 3),
  - Lemma 3 for \( n \equiv 0 \) (mod 3).

c) For \( n \geq 7 \) define the graph \( G_n \in \mathcal{G}_1 \cap \mathcal{G}_2 \) by using the following construction:

- for \( n = 7 \) see the graph \( G_7 \) in Fig. 3; one can easily observe that \( G_7 \in \mathcal{G}_1 \cap \mathcal{G}_2 \);
- having obtained \( G_n \), construct \( G_{n+1} \) using
  - Lemma 1 for \( n \equiv 1 \) (mod 3),
  - Lemma 2 for \( n \equiv 2 \) (mod 3),
  - Lemma 3 for \( n \equiv 0 \) (mod 3).
Fig. 3

References


Souhrn

GRAFY S NEIZOMORFNNÍMI OKOLÍMI UZLŮ 1. A 2. DRHU
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V článku se zkoumá vzájemný vztah tříd \( \Theta_1, \Theta_2 \) grafů s neizomorfními okolími uzlů prvního, resp. druhého druhu; z hlavní věty článku jako důsledek vyplývá, že každá z tříd \( \Theta_1, \Theta_2, \Theta_2 - \Theta_1, \Theta_1 \cap \Theta_2 \) je nekonečná.

Резюме

ГРАФЫ С НЕИЗОМОРФНЫМИ ОКРУЖЕНИЯМИ ВЕРШИН ПЕРВОГО И ВТОРОГО ТИПОВ
ЗДЕНЕК РЬЯЦЕК

В статье изучается взаимоотношение классов \( \Theta_1, \Theta_2 \) графов с неизоморфными окружениями вершин первого и второго типа. Из главной теоремы в качестве следствия вытекает, что каждый из классов \( \Theta_1 - \Theta_2, \Theta_2 - \Theta_1, \Theta_1 \cap \Theta_2 \) бесконечен.

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