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DISTANCES BETWEEN DIRECTED GRAPHS

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Summary. The concepts of the edge distance and of the edge rotation distance introduced by other authors for undirected graphs are adapted for directed graphs.

Keywords: Directed graph, edge distance, edge rotation distance.

AMS Classification: 05C20

In this paper we shall modify the concepts of the edge distance (introduced by V. Kvasnička, V. Baláž, M. Sekanina and J. Koča [2]) and of the edge rotation distance (introduced by G. Chartrand, F. Saba and H.-B. Zou [1]) for directed graphs. We shall consider finite directed graphs without loops in which there exist no two edges with the common initial vertex and with the common terminal vertex.

Let \( \mathcal{G}_1, \mathcal{G}_2 \) be two isomorphism classes of directed graphs, let \( G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2 \). Let \( G_{12} \) be a graph which is isomorphic simultaneously to a subgraph of \( G_1 \) and to a subgraph of \( G_2 \) and has the maximum number of edges from all graphs with this property. Let \( V_1 \) (or \( V_2 \)) be the vertex set of \( G_1 \) (or \( G_2 \), respectively). Let \( E_1 \) (or \( E_2 \), or \( E_{12} \)) be the edge set of \( G_1 \) (or \( G_2 \), or \( G_{12} \), respectively). Then the edge distance of \( G_1 \) and \( G_2 \) is

\[
d_e(\mathcal{G}_1, \mathcal{G}_2) = |E_1| + |E_2| - 2|E_{12}| + ||V_1| - |V_2||.
\]

In the sequel we shall use such pairs \( \mathcal{G}_1, \mathcal{G}_2 \) that \( |V_1| = |V_2| = n, |E_1| = |E_2| = m \). Then we have

\[
d_e(\mathcal{G}_1, \mathcal{G}_2) = 2m - 2|E_{12}|.
\]

In this paper the edge distance will be used only as an auxiliary concept. The main concept will be the edge rotation distance.

As we work with directed graphs, we may introduce two types of the edge rotation. Let \( x, y, z \) be three distinct vertices of a digraph \( G \) such that the edge \( xy \) is in the edge set \( E(G) \) of \( G \), while the edge \( xz \) is not. (We omit arrows over the symbols of edges for typographical reasons.) A rotation of the edge \( xy \) around its initial vertex (shortly \( I \)-rotation) is a transformation of \( G \) by deleting the edge \( xy \) and adding the edge \( xz \). If we suppose that the edge \( yz \) is not in \( E(G) \) (the edge \( xz \) may be), then a rotation of \( xy \) around its terminal vertex (shortly \( T \)-rotation) is a transformation
of $G$ by deleting $xy$ and adding $zy$. An edge rotation is either an $I$-rotation, or a $T$-rotation.

Let $\mathcal{G}_1, \mathcal{G}_2$ be two isomorphism classes of directed graphs such that any graph $G_1 \in \mathcal{G}_1$ has the same number $n$ of vertices and the same number $m$ of edges as any graph $G_2 \in \mathcal{G}_2$. We define the edge rotation distance $d_r(\mathcal{G}_1, \mathcal{G}_2)$ as the minimum number of edge rotations which are necessary for transforming a graph from $\mathcal{G}_1$ into a graph from $\mathcal{G}_2$. Instead of speaking about the distance between isomorphism classes of graphs we shall sometimes speak about the distance between graphs. The distance between the graphs $G_1, G_2$ is the distance between isomorphism classes $\mathcal{G}_1, \mathcal{G}_2$ such that $G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2$.

Now we define the degree vectors of a digraph. Let $G$ be a digraph with $n$ vertices. For $i = 0, 1, \ldots, n - 1$ let $v^+_i(G)$ (or $v^-_i(G)$) be the number of vertices of $G$ of the outdegree (or indegree, respectively) equal to $i$. Now we have two $n$-dimensional vectors

$$v^+(G) = (v^+_0(G), v^+_1(G), \ldots, v^+_n(G)),$$

$$v^-(G) = (v^-_0(G), v^-_1(G), \ldots, v^-_n(G)).$$

The vector $v^+(G)$ (or $v^-(G)$) is called the outdegree (or indegree, respectively) vector of $G$. Both these vectors are called degree vectors of $G$.

**Theorem 1.** Let $G_1, G_2$ be two digraphs with the same number $n$ of vertices and the same number $m$ of edges. Then the following two assertions are equivalent:

(i) $v^+(G_1) = v^+(G_2)$.

(ii) The graph $G_1$ can be transformed into a graph isomorphic to $G_2$ by a sequence of $I$-rotations.

**Proof.** (i) $\Rightarrow$ (ii). If $v^+(G_1) = v^+(G_2)$, we may find a one-to-one correspondence between the vertex sets of $G_1$ and of $G_2$ such that the corresponding vertices have equal outdegrees $G_1$ and in $G_2$, respectively. Now we identify each vertex of $G_1$ with the corresponding vertex of $G_2$; then $G_1$ and $G_2$ may be considered as graphs with the common vertex set. Let $u$ be a vertex of this set; then the set of edges outgoing from $u$, belonging to $G_1$ and not belonging to $G_2$, has the same cardinality as the set of edges outgoing from $u$, belonging to $G_2$ and not belonging to $G_1$. Thus each edge from the first set can be transferred by an $I$-rotation into an edge of the second set. If we do this for all vertices $u$, the graph $G_1$ is transformed into $G_2$.

(ii) $\Rightarrow$ (i). After performing an $I$-rotation, evidently the outdegree vector of the graph remains unchanged; this implies the assertion.

**Theorem 1'.** Let $G_1, G_2$ be two digraphs with the same number $n$ of vertices and the same number $m$ of edges. Then the following two assertions are equivalent:

(i) $v^-(G_1) = v^-(G_2)$.

(ii) The graph $G_1$ can be transformed into $G_2$ by a sequence of $T$-rotations.

**Proof** is dual to the proof of Theorem 1.
If $v^+(G_1) = v^+(G_2)$ (or $v^-(G_1) = v^-(G_2)$) for graphs $G_1, G_2$, we say that $G_1, G_2$ are outdegree equivalent (or indegree equivalent, respectively). Two isomorphism classes $\mathcal{G}_1, \mathcal{G}_2$ of digraphs are called outdegree equivalent (or indegree equivalent, respectively), if so are the graphs $G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2$.

Now we may define the distances $d_{Ir}, d_{Tr}$. If $\mathcal{G}_1, \mathcal{G}_2$ are outdegree (or indegree) equivalent isomorphism classes of digraphs, then their I-rotation distance $d_{Ir}(\mathcal{G}_1, \mathcal{G}_2)$ (or their T-rotation distance $d_{Tr}(\mathcal{G}_1, \mathcal{G}_2)$) is the minimum number of I-rotations (or T-rotations, respectively) which are necessary for transforming a graph from $\mathcal{G}_1$ into a graph from $\mathcal{G}_2$.

The fact that $d_r, d_{Ir}, d_{Tr}$ are metrics can be easily proved analogously as in [1] for undirected graphs.

The following two assertions are evident, because any I-rotation and any T-rotation is an edge rotation.

**Proposition 1.** Let $\mathcal{G}_1, \mathcal{G}_2$ be two outdegree equivalent isomorphism classes of directed graphs. Then

$$d_{Ir}(\mathcal{G}_1, \mathcal{G}_2) \geq d_r(\mathcal{G}_1, \mathcal{G}_2).$$

**Proposition 1'.** Let $\mathcal{G}_1, \mathcal{G}_2$ be two indegree equivalent isomorphism classes of directed graphs. Then

$$d_{Tr}(\mathcal{G}_1, \mathcal{G}_2) \geq d_r(\mathcal{G}_1, \mathcal{G}_2).$$

It might seem that $d_{Ir}$ and $d_{Tr}$, if they are defined, are always equal to $d_r$. But the following theorem shows that it is not so.

**Theorem 2.** There exist outdegree equivalent isomorphism classes $\mathcal{G}_1, \mathcal{G}_2$ of directed graphs such that

$$d_{Ir}(\mathcal{G}_1, \mathcal{G}_2) > d_r(\mathcal{G}_1, \mathcal{G}_2).$$

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**Fig. 1**

Proof. In Fig. 1 we see a graph $G_1 \in \mathcal{G}_1$ and a graph $G_2 \in \mathcal{G}_2$; we consider them...
as graphs with the common vertex set. Evidently
\[ d_r(\mathcal{G}_1, \mathcal{G}_2) = 1, \]
because \( G_2 \) is obtained from \( G_1 \) by a \( T \)-rotation of the edge \( xy \) to the position \( zx \). For the graphs \( G_1, G_2 \) we have
\[ v^+(G_1) = v^+(G_2) = (2, 4, 3, 0, 0, 0, 0, 0) \]
and hence \( d_r(\mathcal{G}_1, \mathcal{G}_2) \) is defined. It is equal to 2, because \( G_2 \) can be obtained from \( G_1 \) by the \( I \)-rotation of \( uz \) into the position \( uy \) and by the \( I \)-rotation of \( vy \) into the position \( vz \), while by one \( I \)-rotation it is evidently not possible.

**Theorem 2'.** There exist indegree equivalent isomorphism classes \( \mathcal{G}_1, \mathcal{G}_2 \) of directed graphs such that
\[ d_{tr}(\mathcal{G}_1, \mathcal{G}_2) > d_r(\mathcal{G}_1, \mathcal{G}_2). \]

**Proof.** These classes are those containing the graphs obtained from the graphs, in Fig. 1 by reversing the orientations of all edges.

Consider again two outdegree equivalent isomorphism classes \( \mathcal{G}_1, \mathcal{G}_2 \) and graphs \( G_1 \in \mathcal{G}_1, G_2 \in \mathcal{G}_2 \) such that \( G_2 \) is obtained from \( G_1 \) by \( d_{tr}(\mathcal{G}_1, \mathcal{G}_2) \) \( I \)-rotations. Thus \( G_1 \) and \( G_2 \) have the common vertex set \( V \). Let \( E_1 \) (or \( E_2 \)) be the edge set of \( G_1 \) (or \( G_2 \), respectively). Denote \( F_0 = E_1 \cap E_2, F_1 = E_1 - E_2, F_2 = E_2 - E_1 \). If an edge \( e_2 \) is obtained from an edge \( e_1 \) by \( I \)-rotations, then evidently it can be obtained from \( e_1 \) by one \( I \)-rotation. Thus \( |F_1| = |F_2| = d_{tr}(\mathcal{G}_1, \mathcal{G}_2) \), because each edge from \( F_2 \) is obtained by an \( I \)-rotation of one edge from \( F_1 \) and the edges from \( F_0 \) may remain unchanged. The graph \( G_0 \) with the vertex set \( V \) and with the edge set \( F_0 \) has \( m - d_{tr}(\mathcal{G}_1, \mathcal{G}_2) \) edges, where \( m \) is the number of edges of \( G_1 \) and of \( G_2 \). The graph \( G_0 \) has the property that it can be embedded into a graph \( G_1 \in \mathcal{G}_1 \) and simultaneously into a graph \( G_2 \in \mathcal{G}_2 \) in such a way that in both the cases each vertex of \( G_0 \) is mapped onto vertices of equal outdegrees in \( G_1 \) and in \( G_2 \). The class of such graphs will be denoted by \( OC(\mathcal{G}_1, \mathcal{G}_2) \). Analogously we define the class \( IC(\mathcal{G}_1, \mathcal{G}_2) \) by replacing the word "outdegrees" by "indegrees".

**Theorem 3.** Let \( \mathcal{G}_1, \mathcal{G}_2 \) be two outdegree equivalent isomorphism classes of directed graphs with \( m \) edges. Then the maximum number of edges of a graph from \( OC(\mathcal{G}_1, \mathcal{G}_2) \) is equal to \( m - d_{tr}(\mathcal{G}_1, \mathcal{G}_2) \).

**Proof.** From the above considerations we have seen that \( G_0 \in OC(\mathcal{G}_1, \mathcal{G}_2) \) and has \( m - d_{tr}(\mathcal{G}_1, \mathcal{G}_2) \) edges. On the other hand, suppose that there exists \( F' \in OC(\mathcal{G}_1, \mathcal{G}_2) \) with \( m' \) edges. We embed \( F' \) into \( G_1 \in \mathcal{G}_1 \) and into \( G_2 \in \mathcal{G}_2 \) and then we choose a one-to-one correspondence between the vertex sets of \( G_1 \) and \( G_2 \), such that if a vertex \( x_1 \) of \( G_1 \) and a vertex \( x_2 \) of \( G_2 \) are images of the same vertex of \( F' \) in both the embeddings, then they correspond to each other. If we identify each pair of the corresponding vertices, the graphs \( G_1, G_2 \) become graphs with the common
vertex set and with the common subgraph $F'$. There are $m - m'$ edges of $G_1$ not belonging to $F'$ and also $m - m'$ edges of $G_2$ not belonging to $F'$. As $v^+(G_1) = v^+(G_2)$, for each vertex $x$ the number of edges outgoing from $x$ which belong to $G_1$ and not to $F'$ is equal to the number of edges adjacent to $x$ which belong to $G_2$ and not to $F'$. Thus each edge belonging to $G_1$ and not to $F'$ either belongs also to $G_2$, or can be transferred by an $I$-rotation into an edge of $G_2$ not belonging to $F'$. As there are $m - m'$ edges belonging to $G_1$ and not to $F'$ there exist $I$-rotations which transform $G_1$ into $G_2$ and whose number is less than or equal to $m - m'$. Thus $d_{tr}(G_1, G_2) \leq m - m'$, which implies

$$m' \leq m - d_{tr}(G_1, G_2).$$

**Theorem 3'.** Let $G_1, G_2$ be two indegree equivalent isomorphism classes of directed graphs with $m$ vertices. Then the maximum number of edges of a graph from $IC(G_1, G_2)$ is equal to $m - d_{tr}(G_1, G_2)$.

**Proof.** Dual to that of Theorem 3.

We shall compare this with the definition of $d_e$. By $C(G_1, G_2)$ we denote the class of all graphs which are isomorphic simultaneously to a subgraph of a graph from $G_1$ and to a subgraph of a graph from $G_2$. Obviously $OC(G_1, G_2) \subseteq C(G_1, G_2)$, $IC(G_1, G_2) \subseteq C(G_1, G_2)$. Thus the maximum number $m_c$ of edges of a graph from $C(G_1, G_2)$ is greater than or equal to the maximum number $m_{oc}$ of edges of a graph from $OC(G_1, G_2)$ and also greater than or equal to the maximum number $m_{IC}$ of edges a graph from $IC(G_1, G_2)$.

We have another theorem.

**Theorem 4.** For two outdegree equivalent isomorphism classes $G_1, G_2$ of directed graphs we have

$$d_{tr}(G_1, G_2) \geq \frac{1}{2}d_e(G_1, G_2).$$

**Proof.** We have

$$d_e(G_1, G_2) = 2m - 2m_c$$

by the definition. Further, Theorem 3 implies that

$$d_{tr}(G_1, G_2) = m - m_{oc}.$$

From these two equalities we obtain the assertion.

**Theorem 4'.** For two indegree equivalent isomorphism classes $G_1, G_2$ of directed graphs we have

$$d_{tr}(G_1, G_2) \geq \frac{1}{2}d_e(G_1, G_2).$$

**Proof.** Dual to the proof of Theorem 4.

Now we shall consider $d_e(G_1, G_2)$. We shall again consider two isomorphism classes $G_1, G_2$ of digraphs with the same number $n$ of vertices and the same number $m$. 

363
of edges. Let $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ and let $C_2$ be obtained from $G_1$ by $d_*(\mathcal{G}_1, \mathcal{G}_2)$ edge rotations. Thus $G_1, G_2$ have a common vertex set $V$. Similarly as above we denote the edge sets of $G_1, G_2$ by $E_1, E_2$, respectively, and further, $F_0 = E_1 \cap E_2$, $F_1 = E_1 - E_2$, $F_2 = E_2 - E_1$. Evidently there exists a mapping $\phi$ of $F_1$ onto $F_2$ such that when transforming $G_1$ into $G_2$ by the minimum number of edge rotations the edge $e \in F_1$ is transferred by edge rotations to $\phi(e) \in F_2$. The mapping $\phi$ is one-to-one. Now we prove a lemma.

**Lemma.** Let $x_1, y_1, x_2, y_2$ be vertices of a digraph $G$ (having at least three vertices) such that the edge $x_1 y_1$ belongs to $G$ and $x_2 y_2$ does not. If $x_2 \neq y_1$ or $y_2 \neq x_1$, then the edge $x_1 y_1$ can be transferred into $x_2 y_2$ by one or two edge rotations. If $x_2 = y_1$ and $y_2 = x_1$, then this can be done by three edge rotations.

**Proof.** If $x_1 = x_2$, the edge $x_1 y_1$ can be transferred into $x_2 y_2$ by one $I$-rotation around $x_1$. If $y_1 = y_2$, this can be done by one $T$-rotation around $y_1$. Let $x_1 \neq x_2$, $y_1 \neq y_2$. Suppose $x_2 \neq y_1$. If $x_2 y_1$ does not belong to $G_2$, we perform the $T$-rotation of $x_1 y_1$ into the position $x_2 x_1$ and then the $I$-rotation of $x_2 y_1$ into the position $x_2 y_2$. If $x_2 y_1$ belongs to $G_2$, we perform first the $I$-rotation of $x_2 y_1$ into $x_2 y_2$ and then the $T$-rotation of $x_1 y_1$ into $x_2 y_2$. If $x_2 = y_1$ but $x_1 \neq y_2$, we may proceed dually. Now the case $x_2 = y_1$, $y_2 = x_1$ remains; then $x_2 y_2$ is oriented inversely to $x_1 y_1$. Choose a vertex $z$ different from $x_1, y_1$. If neither $zy_1$ nor $zx_1$ is in $G$, then we perform a $T$-rotation $R_1$ of $x_1 y_1$ into $zy_1$, an $I$-rotation $R_2$ of $zy_1$ into $zx_1$ and a $T$-rotation $R_3$ of $zx_1$ into $y_1 x_1 = x_2 y_2$. If $zy_1$ is in $G$ and $zx_1$ is not, we perform first $R_2$, then $R_3$ and $R_1$. If both $zy_1$, $zx_2$ are in $G$, we perform the rotations in the order $R_3$, $R_2$, $R_1$. If $zx_1$ is in $G$ and $zy_1$ is not, we perform them in the order $R_1$, $R_3$, $R_2$. On the other hand, we see that two edge rotations are not sufficient, because we do not admit loops.

**Theorem 5.** Let $\mathcal{G}_1, \mathcal{G}_2$ be isomorphism classes of directed graphs with the same number $n$ of vertices and the same number $m$ of edges. If $d_*(\mathcal{G}_1, \mathcal{G}_2) > 2$, then

\[ \frac{1}{2} d_*(\mathcal{G}_1, \mathcal{G}_2) \leq d_*(\mathcal{G}_1, \mathcal{G}_2) \leq d_*(\mathcal{G}_1, \mathcal{G}_2) . \]

If $d_*(\mathcal{G}_1, \mathcal{G}_2) = 2$, then

\[ 1 \leq d_*(\mathcal{G}_1, \mathcal{G}_2) \leq 3 . \]

**Remark.** As we see from (1), the value of $d_*(\mathcal{G}_1, \mathcal{G}_2)$ under the described conditions is always even.

**Proof.** The formula (1) can be rewritten with the new notation:

\[ d_*(\mathcal{G}_1, \mathcal{G}_2) = 2m - 2m_c . \]

Let $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$. Let $G_0 \in C(\mathcal{G}_1, \mathcal{G}_2)$ and let $G_0$ have $m_c$ edges. We perform the same procedure with $G_1, G_2, G_0$ as with $G_1, G_2, F'$ in the proof of Theorem 3. Then $G_1, G_2$ have the common vertex set $V$ and a common subgraph $G_0$. Use again
the notation \( E_1, E_2, F_0, F_1, F_2 \) as above. We have \(|F_1| = |F_2| = m - m_c = \frac{1}{2}d_e(\mathcal{G}_1, \mathcal{G}_2)\). Choose a one-to-one mapping \( \varphi \) of \( F_1 \) onto \( F_2 \). If \(|F_1| \geq 2\), this mapping can be chosen is such a way that no edge is mapped onto the edge obtained from it by reversing the orientation. Now each edge \( e \in F_1 \) will be transformed into its image \( \varphi(e) \in F_2 \) by edge rotations. If \( \varphi(e) \) is not obtained from \( e \) by reversing the orientation for any \( e \), then, according to Lemma, the number of the necessary edge rotations is greater than or equal to \(|F_1|\) and less than or equal to \( 2|F_1| \).

Thus

\[
\frac{1}{2}d_e(\mathcal{G}_1, \mathcal{G}_2) \leq d_\varphi(\mathcal{G}_1, \mathcal{G}_2).
\]

If \(|F_1| = 1\), then the element of \( F_1 \) can be transferred (again according to Lemma) into the element of \( F_2 \) by one, two or three edge rotations, and thus

\[
1 \leq d_\varphi(\mathcal{G}_1, \mathcal{G}_2) \leq 3.
\]

**Theorem 6.** Let \( \alpha, \beta \) be positive integers such that \( \alpha \leq \beta \leq 2\alpha \). Then there exist isomorphism classes \( \mathcal{G}_1, \mathcal{G}_2 \) of digraphs with equal numbers of vertices and equal numbers of edges such that

\[
d_\varphi(\mathcal{G}_1, \mathcal{G}_2) = 2\alpha,
\]

\[
d_\varphi(\mathcal{G}_1, \mathcal{G}_2) = \beta.
\]

**Proof.** First construct the graph \( G_0 \). Its vertex set is \( \{u_1, \ldots, u_{2\alpha}, v_1, \ldots, v_{2\alpha}\} \) and its edge set consists of the edges \( u_iu_j, v_iv_j \) for \( 1 \leq i < j \leq 2\alpha \). (Thus \( G_0 \) consists of two connected components which are acyclic tournaments.) The graph \( G_1 \) contains all vertices and edges of \( G_0 \) and, moreover, the edges \( u_{2i}v_{2i} \) for \( i = 1, \ldots, \alpha \). The graphs \( G_2 \) contains also all vertices and edges of \( G_0 \) and, moreover, the edges \( u_{2i-1}v_{2i-1} \) for \( i = 1, \ldots, \beta - \alpha \) and \( u_{2\alpha}v_{2\alpha} \) for \( i = \beta - \alpha + 1, \ldots, \alpha \). The reader may verify himself that \( G_0 \) is a graph from \( C(\mathcal{G}_1, \mathcal{G}_2) \) (where \( \mathcal{G}_1, \mathcal{G}_2 \) are the isomorphism classes containing \( G_1, G_2 \), respectively) with the maximum number of edges; there are \( \alpha \) edges belonging to \( G_4 \) and not to \( G_0 \), hence

\[
d_\varphi(\mathcal{G}_1, \mathcal{G}_2) = \alpha.
\]

Any edge \( u_{2i}v_{2i} \) for \( i = 1, \ldots, \beta - \alpha \) can be transferred by two edge rotations \( u_{2i-1}v_{2i-1} \) and any edge \( u_{2i}v_{2i} \) for \( i = \beta - \alpha + 1, \ldots, \alpha \) can be transferred by one \( I \)-rotation into \( u_{2i}v_{2i-1} \); the total number of these edge rotations is \( \beta \) and this is evidently the minimum number of edge rotations necessary to transform \( G_1 \) into \( G_2 \).

At the end we shall consider outdegree regular graphs and indegree regular ones. A directed graph is called outdegree regular (or indegree regular), if all of its vertices have equal outdegrees (or indegrees, respectively).

**Theorem 7.** Let \( \mathcal{G}_1, \mathcal{G}_2 \) be two isomorphism classes of outdegree regular directed graphs with the same number \( n \) of vertices and the same number \( m \) of edges. Then
$G_1, G_2$ are outdegree equivalent and
\[ d_e(G_1, G_2) = d_{Ir}(G_1, G_2) = \frac{1}{2} d_e(G_1, G_2). \]

Proof. Let $G_1 \in G_1, G_2 \in G_2$. As $G_1, G_2$ have the same number $n$ of vertices and the same number $m$ of edges and are outdegree regular, all vertices of $G_1$ and all vertices of $G_2$ have the same outdegree $m/n$. From the definitions it is evident that $OC(G_1, G_2) = C(G_1, G_2)$ and thus $m_{OC} = m_e$. As
\[ m_{OC} = m - d_{Ir}(G_1, G_2), \]
\[ d_e(G_1, G_2) = 2m - 2m_e, \]
we have
\[ d_{Ir}(G_1, G_2) = \frac{1}{2} d_e(G_1, G_2). \]

As, according to Proposition 1,
\[ d_r(G_1, G_2) \leq d_{Ir}(G_1, G_2) \]
and according to Theorem 5
\[ \frac{1}{2} d_e(G_1, G_2) \leq d_r(G_1, G_2), \]
we have
\[ d_r(G_1, G_2) = \frac{1}{2} d_e(G_1, G_2). \]

Theorem 7'. Let $G_1, G_2$ be two isomorphism classes of indegree regular directed graphs with the same number $n$ of vertices and the same number $m$ of edges. Then $G_1, G_2$ are indegree equivalent and
\[ d_r(G_1, G_2) = d_{Ir}(G_1, G_2) = \frac{1}{2} d_e(G_1, G_2). \]

Proof is dual to the proof of Theorem 7.

Theorem 8. Let $G_1, G_2$ be two isomorphism classes of directed graphs with the same number $n$ of vertices and the same number $m \neq 0$ of edges. Then
\[ d_e(G_1, G_2) \leq 2m - 2 \]
and this bound cannot be improved.

Proof. If $m \neq 0$, then the graph consisting of two vertices and an edge joining them belongs to $C(G_1, G_2)$ and thus $m_e \geq 1$ and $d_e(G_1, G_2) \leq 2m - 2$. An example of classes $G_1, G_2$ for which $d_e(G_1, G_2) = 2m - 2$ are the classes containing the graph $G_1$ which is a directed path of the length $m$ and the graph $G_2$ which is a star with $m$ edges directed from the center.

Note that for the classes $G_1, G_2$ from the proof of Theorem 8 we have $d_r(G_1, G_2) = m - 1$, because they are classes of indegree regular graphs. This leads us to a conjecture.
Conjecture. Let $G_1, G_2$ be two isomorphism classes of directed graphs with the same number $n$ of vertices and the same number $m$ of edges. Then

$$d_r(G_1, G_2) \leq m - 1.$$ 

References


Souhrn

VZDÁLENOSTI MEZI ORIENTOVANÝMI GRAFY

Bohdan Zelinka

Pojmy hranové vzdálenosti a hranové rotační vzdálenosti zavedené dříve jinými autory pro neorientované grafy jsou modifikovány pro případ orientovaných grafů.

Резюме

РАССТОЯНИЯ МЕЖДУ ОРИЕНТИРОВАННЫМИ ГРАФАМИ

Bohdan Zelinka

Понятия реберного расстояния и реберно-вращательного расстояния, введенные другими авторами для неориентированных графов, здесь применены к ориентированным графам.

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