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ON THE LINEAR CONTROL PROBLEM $\dot{x} = Ax + Bu$

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NOTATIONS AND DEFINITIONS

Let $K$ be a compact subset of $E_k$ ($E_k$ is the $k$-dimensional Euclidean space). $(,)$ is the scalar product in $E_k$. The hyperplane $(\varphi, x) = y$ will be called the support hyperplane of $K$ if $(\varphi, y) \leq y$ for all $y \in K$ and if there is a $z \in K$ such that $(\varphi, z) = y$ (then we write $y = \max_{y \in K} (\varphi, y)$). For any $\varphi \in E_k$, $\varphi \neq 0$ a support hyperplane of $K$ is determined, namely the hyperplane $(\varphi, x) = \max_{y \in K} (\varphi, y)$. The point $p \in K$ will be called an exposed point of $K$ if there is a $\varphi \in E_k$ such that $(\varphi, p) = y$ and $(\varphi, y) < y$ for all $y \in K$, $y \neq p$. The set of all exposed points of $K$ will be denoted by $A(K)$; further $\text{conv } K$ let be the convex hull of $K$ and $\partial K$ the boundary of $K$. For a set $M \subset E_k$, $\overline{M}$ is the closure of $M$ in $E_k$. If $K$ is a convex set then to each point of $\partial K$ there is a support hyperplane which passes through this point. This fact is known in the case that $K$ contains an interior point in $E_k$; if the dimension of $K$ is less than $k$ then the whole set $K$ is contained in any hyperplane of the form $(\varphi, x) = y$, $\varphi \neq 0$. Evidently is $A(K) \subset \partial K$. STRASZEWICZ in [1] proved the following properties of the convex hull and the exposed points:

1. $\text{conv } K = \text{conv } A(K)$; 2. $A(K) = A(\text{conv } K)$; 3. the minimal set (in the sense of inclusion) in the system of all compact sets with the property that their convex hull is $\text{conv } K$ is the set $A(K)$.

In this note we consider the linear control system

\begin{equation}
\frac{dx}{dt} = Ax + Bu
\end{equation}

where $x \in E_n$, $u \in U \subset E_r$, $A$ is an $n \times n$ matrix, $B$ is an $n \times r$ matrix and the set $U \subset E_r$ is compact. We suppose that $T > 0$ is fixed.

The control $u(t) : 0 \leq t \leq T$ will be called admissible if the function $u(t)$ is measurable and $u(t) \in U$ for almost all $t \in \langle 0, T \rangle$. The set of all admissible controls (with values in $U$ for almost all $t \in \langle 0, T \rangle$) is denoted by $\Omega(U)$.  

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**Definition.** The control \( u \in \Omega(U) \) will be called an extremal control corresponding to \( \psi \in E_n, \psi \neq 0 \) if

\[
(e^{-A^t}\psi, Bu(\tau)) = \max_{u \in U} (e^{-A^t}\psi, Bu)
\]

holds for almost all \( \tau \in (0, T) \) (\( A' \) is the transposed matrix to \( A \)).

**Remark.** Each \( \psi \in E_n, \psi \neq 0 \) determines at least one extremal control which corresponds to \( \psi \); this control certainly need not be unique.

In the following we consider the set

\[
A_T(U) = \left\{ y \in E_n, y = \int_0^T e^{-A^t}Bu(\tau) \, d\tau, u \in \Omega(U) \right\}.
\]

By means of the set \( A_T(U) \) we can express the set \( S_T(U) \) of all points in \( E_n \) that can be reached from the point \( x_0 \) in the time \( T \) with some control from \( \Omega(U) \) in the following way:

\[
S_T(U) = e^{AT}(x_0 + A_T(U)).
\]

**PROPERTIES OF THE SET \( A_T(U) \)**

L. W. Neustadt in [2] proved the following

**Proposition 1.** \( A_T(U) \) is convex and compact.

Let us now introduce

**Proposition 2.** Let \( y^* \in A_T(U) \) and let \( (\psi, x) = y, \psi \neq 0 \) be a support hyperplane of \( A_T(U) \) where \( (\psi, y^*) = \gamma \). Then

\[
y^* = \int_0^T e^{-A^t}Bu^*(\tau) \, d\tau
\]

where \( u^* \) is an extremal control corresponding to \( \psi \).

If conversely \( y^* \) is given by (3) where \( u^* \) is an extremal control corresponding to any \( \psi \in E_n, \psi \neq 0 \) then \( y^* \) is a common point of the set \( A_T(U) \) and the support hyperplane of \( A_T(U) \) which is determined by \( \psi \).

**Proof.** Since \( y^* \in A_T(U) \) there is \( y^* = \int_0^T e^{-A^t}Bu^*(\tau) \, d\tau \) with \( u^* \in \Omega(U) \). It holds

\[
(\psi, y^*) = \int_0^T (e^{-A^t}\psi, Bu^*(\tau)) \, d\tau = \gamma = \max_{x \in A_T(U)} (\psi, x).
\]

If (2) is not fulfilled by \( u^*(\tau) \) on any part of \( (0, T) \) with positive measure then \( (\psi, y^*) \) cannot reach its maximal value \( \gamma \) in \( A_T(U) \). Hence \( u^* \) must be an extremal control corresponding to \( \psi \).

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Conversely if $u^*$ is an extremal control corresponding to $\psi$ then $y^*$ given by (3) is contained in $A_T(U)$. For an arbitrary $y \in A_T(U)$ there is $y = \int_0^T e^{-A^*Bu} \, d\tau$ where $u \in \Omega(U)$ and $(e^{-A^*\psi}, Bu(\tau)) \leq (e^{-A^*\psi}, Bu^*(\tau))$ holds for almost all $\tau \in \langle 0, T \rangle$. Hence

$$ (\psi, y) = \int_0^T (e^{-A^*\psi}, Bu(\tau)) \, d\tau \leq \int_0^T (e^{-A^*\psi}, Bu^*(\tau)) \, d\tau = (\psi, y^*) = \gamma $$

and therefore $y^*$ is contained in the hyperplane $(\psi, x) = \gamma$ which supports the set $A_T(U)$.

**Remark.** J. Kurzweil in [3] (cf. Theorem 3 in [3]) proved similarly an analogous statement for the set of all points which can be transferred in to the origin in time less or equal $T$ in the case of a convex set $U$ which contains 0 as its interior point. As for each point of $\partial A_T(U)$ there is at least one support hyperplane of $A_T(U)$ passing through it, it is possible — by Proposition 2 — to express each point of $\partial A_T(U)$ in the form (3) where $u^*$ is some extremal control.

We prove

**Lemma 1.** $A_T(U) = A_T(\text{conv } U)$.

**Proof.** Evidently $A_T(U) \subset A_T(\text{conv } U)$. The converse inclusion will be proved by contradiction. Let exist $y \in A_T(\text{conv } U)$ such that $y \not\in A_T(U)$. By the strict separation theorem for a compact convex set and a closed set (see [4]) there is a $\psi \in E_n$ such that $\gamma = \max \langle \psi, x \rangle < \langle \psi, y \rangle$; $(\psi, x) = \gamma$ is a support hyperplane of $A_T(U)$. We can write $y = \int_0^T e^{-A^*\psi} Bu(\tau) \, d\tau$ where $u \in \Omega(\text{conv } U)$. Further evidently max $(e^{-A^*\psi}, Bu) = \max_{u \in U} \int_0^T e^{-A^*\psi} Bu(\tau) \, d\tau$. We determine $u^* \in \Omega(U)$ such that (2) is fulfilled and write $y^* = \int_0^T e^{-A^*Bu^*} \, d\tau$. Hence $y^* \in A_T(U)$ and $(\psi, y^*) = (\psi, y) > \gamma$. This contradiction gives $A_T(U) \supset A_T(\text{conv } U)$.

From Lemma 1 $A_T(U) = A_T(U_1)$ follows for such $U_1$ that $\text{conv } U_1 = \text{conv } U$ holds. According to results of Straszewicz (see 3. page 141) the minimal compact set with this property is the set $\bar{A}(U)$ therefore $A_T(U) = A_T(\bar{A}(U))$ holds. Hence $S_T(U) = S_T(\bar{A}(U))$, too.

We have the following

**Theorem.** A point which can be reached from the point $x_0 \in E_n$ by any control $u \in \Omega(U)$ in the time $T$ can be reached by a control $u^* \in \Omega(\bar{A}(U))$, too.

**Remark.** This theorem is an analogon of the well known bang-bang principle of LaSalle (see J. P. LaSalle: The time optimal control problem, Contr. to the Theory of Nonlinear Oscillations, Vol. 5), Actually: if $U$ is the unit cube $|u_i| \leq 1$, $i = 1, \ldots, r$ then $\bar{A}(U) = V$ where $V$ are the vertices of the cube $U$. 

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UNIQUE EXTREMAL CONTROLS

We suppose in the following that $B u = 0$ iff $u = 0$. Under this condition the following propositions are known (see [5]):

**Proposition 3.** For almost all $\psi \in E_n$ (in the sense of the $n$-dimensional Lebesgue measure) the extremal control corresponding to $\psi$ is given uniquely almost everywhere in $\langle 0, T \rangle$.

Evidently if $u^*$ is an extremal control which is given uniquely almost everywhere in $\langle 0, T \rangle$ then $u^* \in \Omega(A(U))$ with respect to the property of $B$. Further similarly as in [5] holds

**Proposition 4.** Let the extremal control $u^*$ corresponding to $\psi \in E_n$ be given uniquely almost everywhere in $\langle 0, T \rangle$ and let $y^*$ be given by (3). Then $y^* \in A(A_T(U))$. and also the converse

**Proposition 5.** If $y^* \in A(A_T(U))$ then it is possible to write $y^*$ in the form (3) where $u^*$ is an extremal control which corresponds to some $\psi \in E_n$ and is uniquely determined almost everywhere in $\langle 0, T \rangle$.

**Remark.** Proposition 4 holds even if the above condition for the matrix $B$ is not fulfilled.

Since by the quoted results of [1] is $A_T(U) = \text{conv} A_T(U) = \overline{\text{conv} A(A_T(U))}$ we receive from Propositions 4 and 5 the following

**Theorem.** The set $A_T(U)$ is the convex hull of the closure of all points $y^*$ which can be written in the form (3), with an extremal control uniquely determined almost everywhere in $\langle 0, T \rangle$.

**References**


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