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ON THE SPECTRUM OF MULTIPLIERS IN BESSEL POTENTIAL SPACES

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In this paper we obtain a description of the pointwise, residual and continuous spectra of multipliers in the Bessel potential space $H_p^l(\mathbb{R}^n)$ and its dual $H_{p'}^{-l}(\mathbb{R}^n)$, $p \in (1, \infty)$, $l > 0$ (see [1] for the general properties of these spaces)*). This result is immediately applied to a characterization of the same spectra of a convolution operator in a weighted L_2-space.

By a multiplier in a function space S we mean such a function that multiplication by it maps S into itself. Thus, a space S is associated with the space MS of multipliers. The norm of the multiplication operator in S serves as a norm in MS. Necessary and sufficient conditions for functions to belong to the class MH_p^l are derived in [2].

1. PRELIMINARY INFORMATION

We present certain definitions and simplest facts of the spectral theory.

Let X be a complex Banach space and let A be a bounded linear operator in X.

Definition 1. The set of complex values λ for which the operator $(\lambda I - A)^{-1}$ exists, is defined on the whole of X and is bounded, is called the *resolvent set* $\varrho(A)$ of the operator A. The complement of $\varrho(A)$ is called the *spectrum* $\sigma(A)$ of A.

It is known that the resolvent set $\varrho(A)$ is open and that the function $(\lambda I - A)^{-1}$ is analytic on $\varrho(A)$.

Definition 2. The value

$$r(A) = \sup |\sigma(A)|$$

*) The reference to $\mathbb{R}^n$ in the notation of spaces and norms will be omitted.

is called the *spectral radius* of the operator A.

The Gel'fand formula

$$r(A) = \lim_{m \to \infty} \sqrt[m]{\|A^m\|} \tag{1}$$

is valid.

Definition 3. The operator A is called *quasinilpotent* if

$$\lim_{m \to \infty} \sqrt[m]{\|A^m\|} = 0 .$$

The next three definitions give a classification of points of the spectrum.

Definition 4. The set of numbers $\lambda \in \sigma(A)$ such that the mapping $\lambda I - A$ is not one-to-one is called the *pointwise spectrum* and is denoted by $\sigma_p(A)$. In other words, $\lambda \in \sigma_p(A)$ if and only if there exists a nontrivial solution $u \in X$ of the equation $(\lambda I - A)\, u = 0$. The elements of σ_p are called *eigenvalues.*

Definition 5. The set of numbers $\lambda \in \sigma(A)$ for which the mapping $\lambda I - A$ is one-to-one and the range of $\lambda I - A$ is not dense in X is called the *residual spectrum* and is denoted by $\sigma_r(A)$.

Definition 6. The set of numbers $\lambda \in \sigma(A)$ for which the mapping $\lambda I - A$ is one-to-one and the range of $\lambda I - A$ is dense in X but does not coincide with X is called the *continuous spectrum* of A and is denoted by $\sigma_c(A)$.

It is clear that the sets $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ are disjoint. By the Banach theorem on the isomorphism the condition $(\lambda I - A)\, X \neq X$ in Definition 5 is unnecessary and therefore

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A) . \tag{2}$$

Let A^* be the operator adjoint to A. Definitions 4–6 imply

$$\sigma_r(A) \subset \overline{\sigma_p(A^*)} \subset \sigma_r(A) \cup \sigma_p(A) , \tag{3}$$

where the bar denotes the complex conjunction.

2. THE SPECTRUM OF A MULTIPLIER

We introduce the Bessel potential space H_p^l ($l \geqq 0$, $1 < p < \infty$), obtained by the completion of C_0^∞ with respect to the norm

$$\|u\|_{H_p{}^l} = \|(-\Delta + 1)^{l/2}\, u\|_{L_p} .$$

Let $J_m f$ denote the Bessel potential of order m with density f [1]. It is well known that

$u \in H_p^m$ if and only if $u = J_m f$ with $f \in L_p$. By $H_{p'}^{-l}$, $1/p + 1/p' = 1$, we mean the space conjugate to H_p^l.

Let $H_{p,\mathrm{loc}}^l = \{u\colon u\eta \in H_p^l \text{ for all } \eta \in C_0^\infty\}$.

In the sequel the role of the space X will be played by H_p^l or $H_{p'}^{-l}$ and a multiplier will be considered as an operator A.

Lemma 1. *Let $\gamma \in MH_p^l$ and let σ be a segment of the real axis such that $\gamma(x) \in \sigma$ for almost all $x \in \mathbb{R}^n$. Further, let $k = l - 1$ for an integer l and $k = [l]$ for $\{l\} > 0$. Suppose that $f \in C^{k,1}(\sigma)$, i.e. $f^{(k)}$ satisfies the Lipschitz condition on σ. Then $f(\gamma) \in MH_p^l$ and*

$$\|f(\gamma)\|_{MH_p{}^l} \leqq c \sum_{j=0}^{k+1} \|f^{(j)}; \sigma\|_{L_\infty} \|\gamma\|_{MH_p{}^l}^j .$$

Proof. Let $\{l\} = 0$. The assertion is obvious for $l = 1$. Let it be proved for $l - 1$. For all $u \in C_0^\infty$ we have

$$\|u f(\gamma)\|_{H_p{}^l} \leqq \|f(\gamma)\, \nabla u\|_{H_p{}^{l-1}} + \|u f'(\gamma)\, \nabla\gamma\|_{H_p{}^{l-1}} + \|u f(\gamma)\|_{L_p} . \tag{4}$$

By the induction hypothesis the first summand on the right-hand side does not exceed

$$c\|\nabla u\|_{H_p{}^{l-1}} \sum_{j=0}^{l-1} \|f^{(j)}; \sigma\|_{L_\infty} \|\gamma\|_{MH_p{}^{l-1}}^j .$$

For the same reason the second summand an the right-hand side of (4) is not greater than

$$c\|u\nabla\gamma\|_{H_p{}^{l-1}} \sum_{j=0}^{l-1} \|f^{(j+1)}; \sigma\|_{L_\infty} \|\gamma\|_{MH_p{}^{l-1}}^j .$$

According to [3], the following inequalities hold:

$$\|\nabla\gamma\|_{M(H_p{}^l \to H_p{}^{l-1})} \leqq c\|\gamma\|_{MH_p{}^l}, \quad \|\gamma\|_{MH_p{}^{l-1}} \leqq c\|\gamma\|_{MH_p{}^l} .$$

So the right-hand side in (4) is dominated by

$$c\|u\|_{H_p{}^l} \sum_{j=0}^{l} \|f^{(j)}; \sigma\|_{L_\infty} \|\gamma\|_{MH_p{}^l}^j .$$

The result follows for the integer l.

Let $l \in (0, 1)$. By the Strichartz theorem [4], for all $u \in C_0^\infty$ we have

$$\|u f(\gamma)\|_{H_p{}^l} \leqq c(\|S_l(u f(\gamma))\|_{L_p} + \|u f(\gamma)\|_{L_p}) ,$$

where

$$(S_l v)(x) = \left(\int_0^\infty \left[\int_{|\theta|<1} |v(x + \theta y) - v(x)|\, \mathrm{d}\theta\right]^2 y^{-1-2l}\, \mathrm{d}y\right)^{1/2} .$$

Since

$$S_l(u f(\gamma)) = |u|\, S_l f(\gamma) + \|f(\gamma)\|_{L_\infty} S_l u \leqq$$

$$\leqq |u| \, \|f'; \sigma\|_{L_\infty} S_l\gamma + \|f(\gamma)\|_{L_\infty} S_l u\,,$$

then

$$\|u f(\gamma)\|_{H_p{}^l} \leqq c(\|f'\|_{L_\infty} \|S_l\gamma\|_{M(H_p{}^l \to L_p)} + \|f(\gamma)\|_{L_\infty}) \|u\|_{H_p{}^l}\,.$$

This together with the inequality

$$\|S_l\gamma\|_{M(H_p{}^l \to L_p)} \leqq c\|\gamma\|_{MH_p{}^l}$$

implies the required estimate

$$\|f(\gamma)\|_{MH_p{}^l} \leqq c(\|f'\|_{L_\infty} \|\gamma\|_{MH_p{}^l} + \|f(\gamma)\|_{L_\infty})\,.$$

For any fractional $l > 1$ we should proceed by induction on $[l]$ (cf. the case $\{l\} = 0$).

Corollary 1. *If $\gamma \in MH_p^l$ and $\|\gamma^{-1}\|_{L_\infty} < \infty$ then $\gamma^{-1} \in MH_p^l$ and*

$$\|\gamma^{-1}\|_{MH_p{}^l} \leqq c\|\gamma^{-1}\|_{L_\infty}^{k+2} \|\gamma\|_{MH_p{}^l}^{k+1}\,,$$

where k is the same as in Lemma 1.

The proof immediately follows from Lemma 1 for $f(\gamma) = \gamma^{-1}$ and from the easily derived inequality

$$\|\gamma\|_{MH_p{}^l} \geqq \|\gamma\|_{L_\infty}$$

(see [2]).

Corollary 1 implies the following assertion.

Corollary 2. *A number λ belongs to the spectrum of a multiplier $\gamma \in MH_p^l$ if and only if $(\gamma - \lambda)^{-1} \notin L_\infty$ or, which is equivalent, for any $\varepsilon > 0$ the set $\{x\colon |\gamma(x) - \lambda| < \varepsilon\}$ has a positive n-dimensional measure.*

Since the adjoint operator of $\gamma \in MH_p^l$ is the multiplier $\bar{\gamma}$ in $H_{p'}^{-l}$, Corollary 2 implies

Corollary 3. *A number λ belongs to the spectrum of $\gamma \in MH_{p'}^{-l}$ if and only if $(\gamma - \lambda)^{-1} \notin L_\infty$.*

From Corollaries 2 and 3 we obtain that the spectral radius $r(\gamma)$ of a multiplier γ in H_p^l or $H_{p'}^{-l}$ is equal to $\|\gamma\|_{L_\infty}$. This and (1) imply

$$\lim_{m \to \infty} \sqrt[m]{[\|\gamma\|_{MH_p{}^l}^m]} = \|\gamma\|_{L_\infty}\,.$$

So, the only quasinilpotent multiplier is zero. In other words, the algebra MH_p^l is semisimple.

This is a generalization of the results obtained in [5] for $p = 2$, $2l < 1$.

3. THE STRUCTURE OF THE SPECTRUM OF A MULTIPLIER

The main theorem of the present section contains a description of the decomposition (2) for multipliers in H_p^l and $H_{p'}^{-l}$. Before we pass to its statement we present certain auxiliary definitions and results.

Definition 7. The *capacity* $\operatorname{cap}(e, H_p^l)$ of a compact set $e \subset \mathbb{R}^n$ is the number

$$\operatorname{cap}(e, H_p^l) = \inf\{\|u\|_{H_p^l}^p : u \in C_0^\infty,\ u \geqq 1 \text{ on } e\}.$$

If E is any subset of $\mathbb{R}^n$ then the values

$$\operatorname{cap}_-(E, H_p^l) = \sup\{\operatorname{cap}(e, H_p^l) : e \subset E,\ e \text{ is a compact set}\},$$

$$\operatorname{cap}^-(E, H_p^l) = \inf\{\operatorname{cap}_-(G, H_p^l) : G \supset E,\ G \text{ is an open set}\}$$

are called the *inner* and *outer capacities* of the set E.

Any analytic (in particular, any Borel) subset of $\mathbb{R}^n$ is measurable with respect to the capacity $\operatorname{cap}(\cdot, H_p^l)$, i.e. $\operatorname{cap}^-(E, H_p^l) = \operatorname{cap}_-(E, H_p^l)$ (see [6]).

If the inner and outer capacities of a set E are equal then their value is called the capacity of E and is denoted by $\operatorname{cap}(E, H_p^l)$.

The capacity of a compact set e may be also defined as follows:

$$\operatorname{cap}(e, H_p^l) = \inf\{\|f\|_{L_p}^p : f \in L_p,\ f \geqq 0,\ J_l f \geqq 1 \text{ on } e\}$$

(see [7]).

Let $V_{p,l}\mu$ denote the non-linear Bessel potential of the measure μ, i.e.

$$V_{p,l}\mu = J_l(J_l\mu)^{p'-1}.$$

The following assertion is proved in [6], [7].

Proposition 1. *Let E be a subset of $\mathbb{R}^n$. If $\operatorname{cap}^-(E, H_p^l) < \infty$ then there exists a unique measure μ_E with the properties*

1) $\|J_l\mu_E\|_{L_{p'}}^{p'} = \operatorname{cap}^-(E, H_p^l)$,

2) $V_{p,l}\mu_W \geqq 1$ *(p, l)-quasi everywhere on E, i.e. everywhere on E except a set of zero outer capacity* $\operatorname{cap}^-(\cdot, H_p^l)$,

3) $\operatorname{supp} \mu_E \subset \bar{E}$,

4) $\mu_E(\bar{E}) = \operatorname{cap}^-(E, H_p^l)$,

5) $(V_{p,l}\mu_E)(x) \leqq 1$ *for all* $x \in \operatorname{supp} \mu_E$.

The measure μ_E is called the capacitary measure of the set E and $V_{p,l}\mu_E$ is called the capacitary potential of the set E.

Definition 8. A function u is called *(p, l)-refined* if for any $\varepsilon > 0$ one can find an open set ω such that $\operatorname{cap}(\omega, H_p^l) < \varepsilon$ and u is continuous on $\mathbb{R}^n \setminus \omega$.

For the proofs of the next assertions see [6].

Proposition 2. *For any $u \in H_{p,\mathrm{loc}}^l$ there exists a (p, l)-refined Borel function which coincides with u almost everywhere.*

Proposition 3. *If two (p, l)-refined functions u_1 and u_2 are equal almost everywhere then they are equal (p, l)-quasi everywhere.*

Henceforth in this section all the functions are assumed to be (p, l)-refined and Borel.

The following assertion is proved for an integer l in [8] and for a fractional l in [9] for compacta. The passage to arbitrary sets does not require new arguments.

Proposition 4. *Let $E \subset \mathbb{R}^n$. The capacity $\operatorname{cap}^-(E, H_p^l)$ is equivalent to the set function $\inf\{\|v\|_{H_p^l}^p : v \in M(E)\}$, where $M(E)$ is the collection of (p, l)-refined functions equal to unity (p, l)-quasi everywhere on E and satisfying the inequalities $0 \leqq v \leqq 1$.*

Definition 9. The set $E \subset \mathbb{R}^n$ is called the *set of uniqueness* for the space H_p^l if the conditions $u \in H_p^l$, $u(x) = 0$ for (p, l)-quasi all $x \in \mathbb{R}^n \setminus E$ imply $u = 0$.

The description of the sets of uniqueness for H_p^l is given by Hedberg [10] and Polking [11]. The first result of such a kind for $H_2^{1/2}$ on a circumference is due to Ahlfors and Beurling [12].

Proposition 5 (Hedberg [10]). *Let E be a Borel subset of $\mathbb{R}^n$. The following conditions are equivalent:*

(i) *E is the set of uniqueness for H_p^l;*

(ii) $\operatorname{cap}(G \setminus E, H_p^l) = \operatorname{cap}(G, H_p^l)$ *for any open set G;*

(iii) $\limsup_{\varrho \to 0} \varrho^{-n} \operatorname{cap}(B_\varrho(x) \setminus E, H_p^l) > 0$ *for almost all x, where $B_\varrho(x) = \{y \in \mathbb{R}^n : |y - x| < \varrho\}$.*

If $lp > n$ then E is the set of uniqueness if and only if it has no interior points.

Now we state our main result, i.e. a theorem which gives a characterization of the sets $\sigma_p(\gamma)$, $\sigma_r(\gamma)$ and $\sigma_c(\gamma)$ for a multiplier γ in H_p^l or $H_{p'}^{-l}$.

Theorem. (i) *Let $\gamma \in MH_p^l$ and $\lambda \in \sigma(\gamma)$.*

1. *$\lambda \in \sigma_p(\gamma)$ if and only if the set $Z_\lambda = \{x : \gamma(x) = \lambda\}$ satisfies none of the conditions* (i)–(iii) *of Propositon 5.*

2. *$\lambda \in \sigma_r(\gamma)$ if and only if the set Z_λ satisfies at least one the conditions of Proposition 5 and* $\operatorname{cap}(Z_\lambda, H_p^l) > 0$.

3. *$\lambda \in \sigma_c(\gamma)$ if and only if* $\operatorname{cap}(Z_\lambda, H_p^l) = 0$.

(ii) *Let $\gamma \in MH_{p'}^{-l}$ and $\lambda \in \sigma(\gamma)$.*

1. *$\lambda \in \sigma_p(\gamma)$ if and only if* $\operatorname{cap}(Z_\lambda, H_p^l) > 0$.

2. *$\lambda \in \sigma_c(\gamma)$ if and only if* $\operatorname{cap}(Z_\lambda, H_p^l) = 0$ *(hence the set $\sigma_r(\gamma)$ is empty).*

An obvious corollary of Propositions 2 and 3 is the following assertion.

Lemma 2. *Let γ be a (p, l)-refined function in MH_p^l. The equation $(\gamma - \lambda)u = 0$ has a nontrivial solution in H_p^l if and only if there exists a (p, l)-refined non-zero function in H_p^l vanishing (p, l)-quasi everywhere outside Z_λ.*

This lemma shows that item (i) 1 of the theorem immediately follows from Proposition 5. The proof of the other items of the theorem is given in the next section.

4. PROOF OF THEOREM

Below we shall use the following assertion.

Lemma 3. *Let γ be a (p, l)-refined function in MH_p^l and let $Z_0 = \{x: \gamma(x) = 0\}$. If $\operatorname{cap}(Z_0, H_p^l) = 0$ then the set γH_p^l is dense in H_p^l.*

Proof. Let $f \in C_0^\infty$ and let $N_\tau = \{x \in \operatorname{supp} f: |\gamma(x)| \leqq \tau\}$. By ε we denote a small positive number and by ω we mean an open set with $\operatorname{cap}(\omega, H_p^l) < \varepsilon$ and such that γ is continuous on $\mathbb{R}^n \setminus \omega$. Let G denote a neighbourhood of the set $N_0 \setminus \omega$ with $\operatorname{cap}(G, H_p^l) < \varepsilon$.

We note that $N_\tau \setminus \omega \subset G$ for $\tau > 0$ small enough. In fact, if for any $\tau > 0$ there exists a point $x_\tau \in N_\tau \setminus \omega$ which is not contained in G then, by the continuity of γ outside ω, the limit point x_0 of the family $\{x_\tau\}$ is in $N_0 \setminus \omega$ contrary to the definition of G.

Consequently, $\operatorname{cap}(N_\tau \setminus \omega, H_p^l) < \varepsilon$ for small values of τ and

$$\operatorname{cap}(N_\tau, H_p^l) \leqq \operatorname{cap}(N_\tau \setminus \omega, H_p^l) + \operatorname{cap}(\omega, H_p^l) < 2\varepsilon .$$

Thus, $\operatorname{cap}(N_\tau, H_p^l) \to 0$ as $\tau \to 0$.

By $\{w_\tau\}_{\tau>0}$ we denote the family of functions in $M(N_\tau)$ such that $\lim_{\tau \to 0} \|w_\tau\|_{H_p^l} = 0$ (see Proposition 4). Further, we put

$$u_{\tau,\delta} = (1 - w_\tau)\frac{\bar{\gamma} f}{\gamma\bar{\gamma} + \delta}$$

where $\delta > 0$. Since $(1 - w_\tau) f \in H_p^l$, $\bar{\gamma} \in MH_p^l$, $\gamma\bar{\gamma} \in MH_p^l$ and $\gamma\bar{\gamma} + \delta \geqq \delta$, then $u_{\tau,\delta} \in H_p^l$. We have

$$f - \gamma u_{\tau,\delta} = w_\tau f + \delta(1 - w_\tau) f/(\gamma\bar{\gamma} + \delta) .$$

Let φ be a smooth increasing function on $[0, +\infty)$, $\varphi(0) = \tau^2/4$, $\varphi(t) = t$ for $t > \tau^2/2$. Since $1 - w_\tau = 0$ quasi everywhere on N_τ,

$$f - \gamma u_{\tau,\delta} = w_\tau f + \delta(1 - w_\tau) f/[\varphi(\gamma\bar{\gamma}) + \delta] .$$

Using the inequality

$$\varphi(\gamma\bar{\gamma}) + \delta > \tau^2/4$$

we get from Lemma 1 that the norm $\|[\varphi(\gamma\bar{\gamma}) + \delta]^{-1}\|_{MH_p{}^l}$ is uniformly bounded with respect to δ. Therefore

$$\|f - \gamma u_{\tau,\delta}\|_{H_p{}^l} \leqq \|w_\tau f\|_{H_p{}^l} + \delta\, k(\tau)$$

where $k(\tau)$ does not depend on δ. We put $\delta(\tau) = \tau/k(\tau)$. Then

$$\|f - \gamma u_{\tau,\delta(\tau)}\|_{H_p{}^l} \leqq c\|w_\tau\|_{H_p{}^l} + \tau$$

and so $\gamma u_{\tau,\delta(\tau)} \to f$ as $\tau \to 0$ in H_p^l. Lemma is proved.

In the next three propositions γ is a (p, l)-refined function from MH_p^l.

Proposition 6. *The number λ is contained in the pointwise spectrum of a multiplier γ in $H_{p'}^{-l}$ if and only if* $\operatorname{cap}(Z_\lambda, H_p^l) > 0$.

Proof. *Sufficiency*. Let R be so large that $\operatorname{cap}(Z_\lambda \cap B_R, H_p^l) > 0$ and let μ be the capacity measure of $Z_\lambda \cap B_R$. Note that for any (p, l)-refined function $u \in H_p^l$ we have $u(x)(\gamma(x) - \lambda) = 0$ for (p, l)-quasi all $x \in Z_\lambda \cap B_R$. By Proposition 1 the last equality holds μ-almost everywhere. Therefore $\int u(\gamma - \lambda)\, d\mu = 0$. In other words, $(\gamma - \lambda)\, \mu = 0$. Since

$$\|\mu\|^{p'}_{H_{p'}{}^{-l}} = \|J_l\mu\|^{p'}_{L_{p'}} = \operatorname{cap}(Z_\lambda \cap B_R, H_p^l) < \infty\,,$$

then $\lambda \in \sigma_p(\gamma)$.

Necessity. Let $\lambda \in \sigma_p(\gamma)$. Then there exists a distribution $T \in H_{p'}^{-l}$, $T \neq 0$ such that $(\gamma - \lambda)\, T = 0$. Therefore $(T, (\gamma - \lambda)\, u) = 0$ for all $u \in H_p^l$ and the set $(\gamma - \lambda)\, H_p^l$ is not dense in H_p^l. The result follows by applying Lemma 3.

Proposition 7. *The number λ is contained in the residual spectrum of a multiplier in H_p^l if and only if $\lambda \notin \sigma_p(\gamma)$ and* $\operatorname{cap}(Z_\lambda, H_p^l) > 0$.

Proof. *Sufficiency*. Since $\operatorname{cap}(Z_\lambda, H_p^l) > 0$ then by Proposition 1, $\bar{\lambda}$ is an eigenvalue of the multiplier $\bar{\gamma}$ in $H_{p'}^{-l}$. This and (3) imply $\lambda \in \sigma_r(\gamma) \cup \sigma_p(\gamma) = \sigma_r(\gamma)$.

Necessity. Let $\lambda \in \sigma_r(\gamma)$. By (3), $\bar{\lambda}$ is an eigenvalue of the multiplier $\bar{\gamma}$ in $H_{p'}^{-l}$. So, according to Proposition 1, $\operatorname{cap}(Z_\lambda, H_p^l) > 0$.

Proposition 8. *The multiplier γ in $H_{p'}^{-l}$ has no residual spectrum.*

Proof. Let $\lambda \in \sigma_r(\gamma)$. By virtue of (3), $\bar{\lambda}$ is an eigenvalue of $\bar{\gamma}$ in H_p^l. This and item (i) 1 of Theorem imply $\operatorname{cap}(G \setminus Z_\lambda, H_p^l) < \operatorname{cap}(G, H_p^l)$ for an open set $G \subset \mathbb{R}^n$. Since $\operatorname{cap}(Z_\lambda, H_p^l) > \operatorname{cap}(G, H_p^l) - \operatorname{cap}(G \setminus Z_\lambda, H_p^l)$, then $\operatorname{cap}(Z_\lambda, H_p^l) > 0$. According to Proposition 6 this means that $\lambda \in \sigma_p(\gamma)$. So we arrive at a contradiction. The proposition is proved.

Thus the statements of Theorem concerning the pointwise and residual spectra are proved. The characterization of the continuous spectrum obviously follows from these criteria and the relation (2).

5. THE SPECTRUM OF THE CONVOLUTION OPERATOR

Let $K: u \to k * u$ be a convolution operator with the kernel k. The results of the previous sections in the case $p = 2$ can be interpreted as theorems on the spectrum of K considered as an operator in $L_2((1 + |x|^2)^{l/2})$. Here we have

$$\|u\|_{L_2((1+|x|^2)^{l/2})} = \left(\int |u|^2 (1 + |x|^2)^l \, dx\right)^{1/2} .$$

According to Corollaries 2 and 3, a number λ belongs to the spectrum $\sigma(K)$ of the operator K continuous in $L_2((1 + |x|^2)^{\pm l/2})$, $l \geqq 0$, if and only if $(Fk - \lambda)^{-1} \notin \notin L_\infty$, where F is the Fourier transform in $\mathbb{R}^n$.

The spectral radius $r(K)$ of the operator K in $L_2((1 + |x|^2)^{\pm l/2})$ is equal to $\|Fk\|_{L_\infty}$.

Let $\lambda \in \sigma(K)$. By the theorem formulated in Sec. 3, λ is an eigenvalue of K if and only if, for all x in a set of positive measure,

$$\lim_{\varrho \to 0} \varrho^{-n} \operatorname{cap}(B_\varrho(x) \setminus Z_\lambda, H_2^l) = 0 ,$$

where $Z_\lambda = \{\xi \in \mathbb{R}^n : (Fk)(\xi) = \lambda\}$. This condition is equivalent to

$$\operatorname{cap}(G \setminus Z_\lambda, H_2^l) < \operatorname{cap}(G, H_2^l)$$

for an open set G. According to the same theorem, λ belongs to the residual spectrum $\sigma_r(K)$ if and only if none of the conditions stated above is valid and $\operatorname{cap}(Z_\lambda, H_2^l) > 0$. Moreover, $\lambda \in \sigma_c(K) \Leftrightarrow \operatorname{cap}(Z_\lambda, H_2^l) = 0$. If λ is a point of the spectrum of an operator K continuous in $L_2((1 + |x|^2)^{-l/2})$, $l > 0$, then our theorem yields that $\lambda \in \sigma_p(K)$ is equivalent to $\operatorname{cap}(Z_\lambda, H_2^l) > 0$. Besides, $\lambda \in \sigma_c(K)$ if and only if $\operatorname{cap}(Z_\lambda, H_2^l) = 0$. Consequently, $\sigma_r(K) = \emptyset$.

References

[1] *A. Kufner, O. John, S. Fučík:* Function spaces. Academia, Prague, 1977.

[2] *V. G. Maz'ya, T. O. Shaposhnikova:* Multipliers in pairs of potential spaces. Math. Nachr. *99* (1981), 363—379.

[3] *V. G. Maz'ya, T. O. Shaposhnikova:* On multipliers in Sobolev spaces (Russian). Vestnik Leningrad. Univ. 7 (1979), 33—40.

[4] *R. S. Strichartz:* Multipliers on fractional Sobolev spaces. J. Math. and Mech. *16* (1967), 1031—1060.

[5] *A. Devinatz, I. I. Hirschman:* Multiplier transformations on $l^{2,\alpha}$. Annals of Math. *69* (1959), 575—587.

[6] *V. G. Maz'ya, V. P. Havin:* Non-linear potential theory (Russian). Uspehi Matem. Nauk *27* (1972), 67—138.

[7] *N. G. Meyers:* A theory of capacities for potentials of functions in Lebesgue classes. Math. Scand. *26* (1970), 255—292.

[8] *V. G. Maz'ya:* On removable singularities of bounded solutions of quasilinear elliptic equations of arbitrary order (Russian). Zap. Nauchn. Semin. Leningr. Otd. Mat. Inst. Steklova *26* (1972), 116—130.

[9] *D. R. Adams, J. C. Polking:* The equivalence of two definitions of capacity. Proc. Amer. Math. Soc. *37* (1973), 529—534.

[10] *L. I. Hedberg:* Approximation in the mean by solutions of elliptic equations. Duke Math. Journ. *40* (1973), 9—16.

[11] *J. C. Polking:* Approximation in L^p by solutions of elliptic partial differential equations. Amer. Math. Journ. *94* (1972), 1231—1244.

[12] *L. Ahlfors, A. Beurling:* Conformal invariants and function-theoretic null-sets. Acta Math. *83* (1950), 101—129.

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