Ivan Chajda; Vítězslav Novák

On extensions of cyclic orders

Časopis pro pěstování matematiky, Vol. 110 (1985), No. 2, 116--121

Persistent URL: `http://dml.cz/dmlcz/108597`

Časopis pro pěstování matematiky, roč. 110 (1985), Praha

ON EXTENSIONS OF CYCLIC ORDERS

IVAN CHAJDA, Přerov, VÍTĚZSLAV NOVÁK, Brno
(Received February 15, 1982)

It is known that not any cyclic order has a linear extension ([3]). In this note we derive some simple sufficient conditions for existence of such an extension.

1. TERNARY RELATIONS

1.1. Definition. Let G be a set. *A ternary relation* T on the set G is any subset of the 3^{rd} cartesian power G^3: $T \subseteq G^3$.

1.2. Definition. Let G be a set, T a ternary relation on G. This relation is called:

(1) *symmetric*, iff $(x, y, z) \in T \Rightarrow (z, y, x) \in T$;
(2) *strongly symmetric*, iff $(x, y, z) \in T \Rightarrow (u, v, w) \in T$ for any permutation (u, v, w) of the sequence (x, y, z);
(3) *asymmetric*, iff $(x, y, z) \in T \Rightarrow (z, y, x) \overline{\in} T$;
(4) *strongly asymmetric*, iff $(x, y, z) \in T \Rightarrow (u, v, w) \overline{\in} T$ for any odd permutation (u, v, w) of the sequence (x, y, z);
(5) *reflexive*, iff $x, y, z \in G$, card $\{x, y, z\} \leqq 2 \Rightarrow (x, y, z) \in T$;
(6) *transitive*, iff $(x, y, z) \in T$, $(x, z, u) \in T \Rightarrow (x, y, u) \in T$;
(7) *cyclic*, iff $(x, y, z) \in T \Rightarrow (y, z, x) \in T$;
(8) *complete*, iff $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow$ there exists a permutation (u, v, w) of the sequence (x, y, z) such that $(u, v, w) \in T$.

1.3. Lemma. *Let G be a set, T a ternary relation on the set G. Then*

(1) *T is strongly symmetric if and only if T is symmetric and cyclic.*
(2) *Let T be cyclic. Then T is strongly asymmetric, if and only if it is asymmetric.*

Proof. (1) is trivial.

(2): Let T be cyclic. If T is strongly asymmetric, then it is obviously asymmetric. If T is asymmetric and $(x, y, z) \in T$, then $(y, z, x) \in T$, $(z, x, y) \in T$, thus $(z, y, x) \overline{\in} T$, $(x, z, y) \overline{\in} T$, $(y, x, z) \overline{\in} T$. Hence T is strongly asymmetric.

1.4. Definition. Let G be a set, T a ternary relation on the set G. The ternary relation T^* on G defined by

$$(x, y, z) \in T^* \Leftrightarrow (z, y, x) \in T$$

is called the *dual relation* to T.

1.5. Lemma. *Let T be a ternary relation on a set G. Let* (p) *be any one of the properties* (1)–(5), (7), (8) *from* 1.2. *If T has the property* (p), *then T^* has the property* (p) *as well.*

Proof is trivial in all cases. We show, for instance, that strong asymmetry of T implies strong asymmetry of T^*. Thus, let T be strongly asymmetric, $(x, y, z) \in T^*$ and let (u, v, w) be an odd permutation of the sequence (x, y, z). Then $(z, y, x) \in T$ and (w, v, u) is an odd permutation of (z, y, x). Thus $(w, v, u) \bar{\in} T$ and $(u, v, w) \bar{\in} T^*$.

1.6. Definition. Let T be a ternary relation on a set G and let (p) be any one of the properties (1)–(8) from 1.2. A ternary relation Q on G is called a (p) – *hull* of the relation T, if and only if

(1) $Q \supseteq T$,
(2) Q has the property (p),
(3) if R is any ternary relation on G having the property (p) and such that $R \supseteq T$, then $R \supseteq Q$.

1.7. Lemma. *Let G be a set, $I \neq \emptyset$ a set, and let T_i be a ternary relation on G for any $i \in I$. Let* (p) *be any one of the properties* (1)–(7) *from* 1.2. *If T_i has the property* (p) *for any $i \in I$ then $T = \bigcap_{i \in I} T_i$ has the property* (p).

Proof is trivial.

1.8. Corollary. *Let T be a ternary relation on a set G and let* (p) *be any one of the properties* (1), (2), (5)–(7) *from* 1.2. *Then there exists a* (p) – *hull of the relation T on G.*

Proof follows from 1.7 and from the fact that the full relation G^3 has the properties (1), (2), (5)–(7).

1.9. Lemma. *Let T be a ternary relation on a set G and let T^s be the symmetric hull of T. Then $T^s = T \cup T^*$.*

Proof. Trivial.

1.10. Lemma. *Let T be a ternary relation on a set G and let T^c be the cyclic hull of T. Then $T^c = \{(x, y, z) \in G^3$; there exists an even permutation (u, v, w) of the sequence (x, y, z) such that $(u, v, w) \in T\}$.*

Proof. Trivial.

1.11. Lemma. *Let T be a ternary relation on a set G and let T^σ be the strongly symmetric hull of T. Then $T^\sigma = (T^s)^c$.*

Proof. Obviously $(T^s)^c \supseteq T$, and $(T^s)^c$ is strongly symmetric. If R is a strongly symmetric ternary relation on G and $R \supseteq T$, it can be easily seen that $(T^s)^c \subseteq R$.

1.12. Notation. Let G be a set. We put $I_G = \{(x, y, z) \in G^3;\ \text{card}\,\{x, y, z\} \leqq 2\}$.

1.13. Lemma. *Let T be a ternary relation on a set G and let T^r be the reflexive hull of T. Then $T^r = T \cup I_G$.*

Proof. Trivial.

1.14. Notation. Let T be a ternary relation on a set G. Put $T^1 = \{(x, y, z) \in G^3$; there exists $u \in G$ such that $(x, y, u) \in T$, $(x, u, z) \in T\}$, $T' = T \cup T^1$. Further we define by induction $T^{(0)} = T$, $T^{(n+1)} = (T^{(n)})'$ for any natural number n.

1.15. Theorem. *Let T be a ternary relation on a set G and let T^t be the transitive hull of T. Then $T^t = \bigcup_{n=0}^{\infty} T^{(n)}$.*

Proof. Denote $\bigcup_{n=0}^{\infty} T^{(n)} = Q$. Obviously $Q \supseteq T$. We show that Q is transitive. Let $(x, y, z) \in Q$, $(x, z, u) \in Q$. Then there exist natural numbers m, n such that $(x, y, z) \in T^{(m)}$, $(x, z, u) \in T^{(n)}$. Put $k = \max\{m, n\}$; then $T^{(m)} \subseteq T^{(k)}$, $T^{(n)} \subseteq T^{(k)}$ and thus $(x, y, z) \in T^{(k)}$, $(x, z, u) \in T^{(k)}$. This implies $(x, y, u) \in (T^{(k)})' = T^{(k+1)}$ and hence $(x, y, u) \in Q$. Thus Q is transitive. Let R be any transitive ternary relation on G such that $R \supseteq T$. We prove by induction that $T^{(n)} \subseteq R$ for any natural n. For $n = 0$ this condition holds by the assumption. Suppose $T^{(m)} \subseteq R$ and $(x, y, z) \in T^{(m+1)} = (T^{(m)})'$. Then either $(x, y, z) \in T^{(m)}$ or there exists $u \in G$ with $(x, y, u) \in T^{(m)}$, $(x, u, z) \in T^{(m)}$. This implies $(x, y, u) \in R$, $(x, u, z) \in R$ and as R is transitive, we have $(x, y, z) \in R$. Hence $Q \subseteq R$.

1.16. Lemma. *Let T be a ternary relation on a set G. If T is strongly asymmetric, then T^c is asymmetric.*

Proof. Let $(x, y, z) \in T^c$, $(z, y, x) \in T^c$. Then there exists an even permutation (u, v, w) of (x, y, z) such that $(u, v, w) \in T$ and an even permutation (r, s, t) of

(z, y, x) such that $(r, s, t) \in T$. As (z, y, x) is an odd permutation of (x, y, z), (r, s, t) is an odd permutation of (u, v, w). But this contradicts the strong asymmetry of T.

2. CYCLICALLY ORDERED SETS

2.1. Definition. Let G be a set, C a ternary relation on the set G which is asymmetric, transitive and cyclic. Then C is called a *cyclic order* on G and the pair (G, C) is called a *cyclically ordered set*. If, moreover, card $G \geqq 3$ and C is complete, then C is called a *complete (linear) cyclic order* on G and (G, C) is called *completely (linearly) cyclically ordered set* or a *cycle*.

In what follows, we summarize some concepts and assertions concerning cyclically ordered sets which can be found in [5].

2.2. *Let G be a set, T a cyclic ternary relation on G. T is transitive if and only if one of the following equivalent conditions holds:*

(1) $(x, y, z) \in T,\ (x, u, y) \in T \Rightarrow (u, y, z) \in T$;
(2) $(x, y, z) \in T,\ (y, u, z) \in T \Rightarrow (x, y, u) \in T$;
(3) $(x, y, z) \in T,\ (y, u, z) \in T \Rightarrow (x, u, z) \in T$.

2.3. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. For any $x, y \in G$ put $x <_{C,x_0} y$ iff either $(x_0, x, y) \in C$ or $x_0 = x \neq y$. Then $<_{C,x_0}$ is an order on G with the least element x_0.*

2.4. *Let G be a set, let $<$ be an order on G. Define the ternary relation $C_<$ on G by $(x, y, z) \in C_< \Leftrightarrow$ either $x < y < z$, or $y < z < x$, or $z < x < y$. Then $C_<$ is a cyclic order on G.*

2.5. *Let G be a set, let $<_1, <_2$ be orders on G. If $<_1 \subseteq <_2$, then $C_{<_1} \subseteq C_{<_2}$.*

2.6. *Let G be a set with* card $G \geqq 3$, *let $<$ be a linear order on G. Then $C_<$ is a linear cyclic order on G.*

Let (G, C) be a cyclically ordered set, let $x_0 \in G$. (G, C) is called x_0*-stable* iff the following condition is fulfilled: $x, y \in G - \{x_0\}$, $(z, x, y) \in C$ for some $z \in G \Rightarrow$ $\Rightarrow (x_0, x, y) \in C$ or $(x_0, y, x) \in C$.

2.7. *Let (G, C) be a cyclically ordered set, let $x_0 \in G$. Then the following statements are equivalent:*

(A) $C = C_{<_{C,x_0}}$,
(B) (G, C) *is* x_0 *– stable.*

Let (G, C) be a cyclically ordered set, let $A \subseteq G$, $A \neq \emptyset$. The subset A is called *connected*, iff the following condition is fulfilled: $x, y \in A$, $x \neq y \Rightarrow$ there exist a natural number n and elements $x_i, y_i, z_i \in A$ $(1 \leqq i \leqq n)$ such that $(x_i, y_i, z_i) \in C$ for all $i = 1, \ldots, n$, $x \in \{x_1, y_1, z_1\}$, $y \in \{x_n, y_n, z_n\}$, and $\{x_i, y_i, z_i\} \cap \{x_{i+1}, y_{i+1}, z_{i+1}\} \neq \emptyset$ for $i = 1, \ldots, n - 1$.

2.8. *Let (G, C) be a cyclically ordered set, let $x \in G$. Then there exists a maximal connected subset of G containing x.*

A maximal connected subset of a cyclically ordered set (G, C) is called a *component* of (G, C).

Let I be a set and let (G_i, C_i) be a cyclically ordered set for any $i \in I$. Let the sets G_i $(i \in I)$ be pairwise disjoint. Put $G = \bigcup_{i \in I} G_i$, $C = \bigcup_{i \in I} C_i$. Then (G, C) is called the *direct sum* of cyclically ordered sets (G_i, C_i) $(i \in I)$; we write $(G, C) = \sum_{i \in I} (G_i, C_i)$.

It is clear that $\sum_{i \in I} (G_i, C_i)$ is a cyclically ordered set. Further, if (G, C) is a cyclically ordered set, $\{G_i;\ i \in I\}$ the set of all its components and $C_i = C \cap G_i^3$ for all $i \in I$, then $(G, C) = \sum_{i \in I} (G_i, C_i)$. This expression is called the *canonical representation* of (G, C).

3. LINEAR EXTENSION OF A CYCLIC ORDER

3.1. Definition. Let G be a set, let C_1, C_2 be cyclic orders on G. C_2 is called an *extension* of C_1 iff $C_1 \subseteq C_2$. An extension C_2 of a cyclic order C_1 on a set G is called a *linear extension* iff C_2 is a linear cyclic order on G.

3.2. Remark. In contrast to the well-known Szpilrajn's theorem on orders ([6]), not every cyclic order has a linear extension. The following example can be found in [3].

3.3. Example. Put $G = \{x_0, y_0, z_0, x, y, z, u, v, w, q, r, s, t\}$, $T = \{(x_0, z_0, x), (y_0, x, y), (z_0, y, z), (x, z, u), (y, u, v), (z, v, x_0), (u, x_0, z_0), (v, z_0, y_0), (x_0, y_0, w), (z_0, w, q), (y_0, q, r), (w, r, s), (q, s, t), (r, t, x_0), (s, x_0, y_0), (t, y_0, z_0), (v, z_0, t), (y_0, v, t)\}$. As T is strongly asymmetric, the cyclic hull T^c of T is asymmetric by 1.16. Further, T^c is cyclic and by a direct verification we find that it is transitive. Thus T^c is a cyclic order on G. Let C be any extension of T^c on G and suppose $(x_0, y_0, z_0) \in C$. Then $(x_0, z_0, x) \in T \subseteq C$ implies $(y_0, z_0, x) \in C$ by 2.2 (1). Analogously $(y_0, x, y) \in T \subseteq C$ implies $(z_0, x, y) \in C$ and we get successively $(x, y, z) \in C$, $(y, z, u) \in C$, $(z, u, v) \in C$, $(u, v, x_0) \in C$, $(v, x_0, z_0) \in C$, $(x_0, z_0, y_0) \in C$. This contradicts the assumption $(x_0, y_0, z_0) \in C$. If we suppose $(x_0, z_0, y_0) \in C$ then we similarly obtain $(z_0, y_0, w) \in C$, $(y_0, w, q) \in C$, $(w, q, r) \in C$, $(q, r, s) \in C$, $(r, s, t) \in C$, $(s, t, x_0) \in C$, $(t, x_0, y_0) \in C$, $(x_0, y_0, z_0) \in C$, a contradiction. Thus, T^c has no linear extension.

3.4. Theorem. *Let* (G, C) *be a cyclically ordered set with* card $G \geqq 3$. *If* (G, C) *is* x_0 — *stable for some* $x_0 \in G$, *then there exists a linear extension of the cyclic order* C *on* G.

Proof. Let (G, C) be x_0-stable. By 2.3, $<_{C,x_0}$ is an order on G. By Szpilrajn's theorem ([6]) there exists a linear extension of the order $<_{C,x_0}$ on G, i.e. there exists a linear order $<$ on G such that $<_{C,x_0} \subseteq <$. By 2.6, $C_<$ is a linear cyclic order on G and by 2.5, $C_{<_{C,x_0}} \subseteq C_<$. But $C_{<_{C,x_0}} = C$ by 2.9, thus $C \subseteq C_<$ and $C_<$ is a linear extension of C.

3.5. Theorem. *Let* (G, C) *be a cyclically ordered set with* card $G \geqq 3$, let $(G, C) = \sum_{i \in I} (G_i, C_i)$ *be its canonical representation. If, for any* $i \in I$, *there exists* $x_i \in G_i$ *such that* (G_i, C_i) *is* x_i-*stable, then* C *has a linear extension on* G.

Proof. $<_{C_i,x_i}$ is an order on G_i for any $i \in I$ by 2.3. As the sets G_i are pairwise disjoint, $<_C = \bigcup_{i \in I} <_{C_i,x_i}$ is an order on G (in fact, $<_C$ is the cardinal sum of orders $<_{C_i,x_i}$). According to Szpilrajn's theorem ([6]) there exists a linear extension $<$ o, the order $<_C$ on G. Thus, we have $<_{C_i,x_i} \subseteq <_C \subseteq <$ for any $i \in I$ and by 2.5t $C_{<_{C_i,x_i}} \subseteq C_{<_C} \subseteq C_<$. As (G_i, C_i) is x_i-stable, 2.9 implies $C_{<_{C_i,x_i}} = C_i$ so thaf $C_i \subseteq C_<$ for any $i \in I$. Hence $\bigcup_{i \in I} C_i = C \subseteq C_<$. But $C_<$ is a linear cyclic order on G by 2.6 and, therefore, $C_<$ is a linear extension of C.

3.6. Remark. The following problem remains open: Find necessary and sufficient conditions for a cyclic order to have a linear extension.

Bibliography

[1] *G. Birkhoff:* Generalized arithmetic. Duke Math. Journ. *9* (1942), 283—302.
[2] *E. Čech:* Bodové množiny (Point sets). Academia Praha, 1966.
[3] *N. Megiddo:* Partial and complete cyclic orders. Bull. Am. Math. Soc. *82* (1976), 274—276.
[4] *G. Müller:* Lineare und zyklische Ordnung. Praxis Math. *16* (1974), 261—269.
[5] *V. Novák:* Cyclically ordered sets. Czech. Math. Journ. *32* (107) (1982), 460—473.
[6] *E. Szpilrajn:* Sur l'extension de l'ordre partiel. Fund. Math. *16* (1930), 386—389.

Address of authors: Ivan Chajda, 750 00 Přerov, Lidových milicí 22. Vítězslav Novák, 662 95 Brno, Janáčkovo nám. 2a (Přírodovědecká fakulta UJEP).